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Edited by T. W. CHAUNDY, U. S. HASLAM-JONES, E. C. THOMPSON

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THE FACTORIZATION OF ABELIAN GROUPS

By A. D. SANDS (*Ibadan*)

[Received 12 December 1957]

1. Introduction

THE problem of the factorization of finite abelian groups has been considered by several authors. In (**1**) de Bruijn gives a list of these groups for which Hajós's problem has not been solved. In (**4**) the results on the cyclic groups given there were presented. In this paper the results for the non-cyclic groups whose expression as a direct product, as given by de Bruijn, does not involve an arbitrary prime are given. The notations, definitions, and preliminary restatements of the problem of (**4**) are used throughout the paper.

In § 2 a theorem on bad groups, similar to the theorems of de Bruijn in (**1**), is proved and this theorem is used to show that the groups of type $\{p^\lambda, 2, 2\}$ and $\{2^\lambda, 2, 2\}$ are bad if $\lambda \geqslant 4$. In § 3 the remaining groups listed by de Bruijn of the types given above are shown to be good. In § 4, I deal with extensions of results to certain infinite abelian groups. The groups considered are those of type $\{p^\infty\}$ and direct products of them with finite abelian groups. It is shown that where an arbitrary integer λ occurred in previous work it can be replaced by ∞. But the restriction is made throughout that the number of elements in one of the factors is finite.

2. Bad groups

In (**1**) de Bruijn proves a set of theorems which give conditions for a group to be bad. In this section I prove formally the fundamental result used by de Bruijn and then give one new theorem similar to the theorems of de Bruijn. This theorem enables us to show that the groups of type $\{p^\lambda, 2, 2\}$ are bad, where p is a prime and $\lambda \geqslant 4$.

THEOREM 1. *If a group G possesses a proper subgroup H and H admits of factorizations $H = AB = AC$, where A is not periodic and B and C have no period in common, then G is bad.*

Proof. Let $k_1, k_2,\ldots, k_n$ be a set of coset representatives for G by H. Let

$$D = Bk_1+Ck_2+Ck_3+\ldots+Ck_n.$$

Then

$$AD = ABk_1 + ACk_2 + ... + ACk_n = Hk_1 + Hk_2 + ... + Hk_n = G.$$

Now A is not periodic. Let g be a period of D. Then $g = hk_i$, where h is in H. Consider $hk_i Bk_1$. Now $k_i k_1 = h_1 k_j$ for some element h_1 in H. Therefore, if b is in B and so in H,

$$hk_i bk_1 = h_2 k_j$$

for some element h_2 in H. Therefore $hk_i Bk_1 \subseteq Hk_j$. But $hk_i D = D$. It follows that

$$hk_i Bk_1 \subseteq Bk_1 + Ck_2 + ... + Ck_n.$$

Thus, if $j = 1$, $hk_i Bk_1 = Bk_1$ and, if $j > 1$, $hk_i Bk_1 = Ck_j$.

In the second case it follows that $C = hk_i k_1 k_j^{-1} B$ and thus that any period of B is a period of C. Since B and C have no period in common, it follows that B and C must be non-periodic. Therefore H is bad and so, by Theorem 4 of (**1**) [263], G is also bad. In the first case $hk_i Bk_1 = Bk_1$. Therefore $hk_i B = B$. It follows that hk_i is in H. Therefore $hk_i Ck_2 \subseteq Hk_2$. But

$$hk_i Ck_2 \subseteq Bk_1 + Ck_2 + ... + Ck_n.$$

Therefore $hk_i Ck_2 = Ck_2$, and so $hk_i C = C$. It follows that hk_i is a period of both B and C. Since B and C have no common period, it follows that D is not periodic. Thus $AD = G$ is a factorization of G with both factors non-periodic. Therefore G is bad.

THEOREM 2. *If a group G possesses a proper subgroup K and K a proper subgroup H which is the direct product of a subgroup L of composite order and a subgroup of type $\{2, 2\}$, then G is bad.*

Proof. It may be assumed that L is not of type $\{2, 2\}$, since in (**1**) [262] de Bruijn has already dealt with this case. Then, by Lemma 1 of (**1**) [259], L contains a proper subgroup M, of order greater than 1, with a set of coset representatives $l_1, l_2, ..., l_k$ of L by M which is not periodic. We assume that l_1 is in M. Let $k_1, k_2, ..., k_n$ be any set of coset representatives for K by H. Let b and c be elements of order 2 generating the subgroup of type $\{2, 2\}$.

Let

$$A = (k_2, ..., k_n).\{(e, bc) + (b, c)(M - e)\} + k_1 M.(e, l_2 bc),$$

where $M - e$ indicates all elements of M except e. Let

$$B = (e, b).(l_1, l_2, ..., l_k), \qquad C = (e, c).(l_1, l_2, ..., l_k).$$

Then

$$\begin{aligned}AB &= (k_2,\dots,k_n)(e,bc)(e)(e,b)(l_1,l_2,\dots,l_k)+\\ &\quad+(k_2,\dots,k_n)(b,c)(M-e)(e,b)(l_1,l_2,\dots,l_k)+\\ &\quad+k_1 M(e,l_2 bc)(e,b)(l_1,l_2,\dots,l_k)\\ &= (k_2,\dots,k_n)(e,b,c,bc)M(l_1,l_2,\dots,l_k)+k_1 L(e,b,l_2 c,l_2,bc)\\ &= (k_2,\dots,k_n)(e,b,c,bc)L+k_1 L(e,b,c,bc)\\ &= (k_1,k_2,\dots,k_n)H = K.\end{aligned}$$

Similarly it can be shown that $AC = K$.

It is easily verified that b is the only period of B and that c is the only period of C. Thus B and C have no period in common.

Let f be a period of A. Then f is an element of K. Therefore multiplication by f will permute the cosets Hk_1, $Hk_2,\dots, Hk_n$ among themselves. If

$$fk_1 M(e,l_2 bc) = k_1 M(e,l_2 bc),$$

then

$$fM(e,l_2 bc) = M(e,l_2 bc),$$

and f is an element of H. Thus f is of one of the forms l, lb, lc, lbc, where l is in L. Clearly f can be only of the first or last form. But, since f is in H, f must also take $\{e, bc+(b,c)(M-e)\}$ into itself. Now l does not do this unless $l = e$ since otherwise $le = l$ does not lie in the set. Similarly lbc does not do this unless $l = e$. But bc is not a period of $M(e,l_2 bc)$ since l_2 is not in M.

The remaining possibility is that f take $k_1 M(e,l_2 bc)$ into

$$k_i\{(e,bc)+(b,c)(M-e)\},$$

where $i > 1$. Now f is of one of the forms

$$k_j l,\quad k_j lb,\quad k_j lc,\quad k_j lbc,$$

where l is an element of L. Let $f_1 = k_j l$. Then both $f_1 k_1$ and $f_1 k_1 l_2$ lie in $k_i M$. It follows that l_2 lies in M. But this is not the case. Therefore A is not periodic.

It follows by Theorem 1 that G is bad.

THEOREM 3. *Groups of type* $\{p^\lambda, 2, 2\}$, *including those of type* $\{2^\lambda, 2, 2\}$, *where* p *is a prime, are bad if* $\lambda \geqslant 4$.

Proof. This follows from Theorem 2, by taking K of type $\{p^3, 2, 2\}$, H of type $\{p^2, 2, 2\}$, and L of type $\{p^2\}$.

These are the only groups to which this theorem applies but to which one of the similar theorems of de Bruijn does not already apply.

3. Good groups

In this section it is shown that the groups of types $\{3^2, 3\}$, $\{2, 3, 3\}$, $\{2^\lambda, 2\}$, $\{2^3, 2, 2\}$, $\{2^2, 2, 2, 2\}$, $\{2^2, 2^2\}$, $\{2^2, 2, 2\}$ are good.

THEOREM 4. *The groups of types* $\{3^2, 3\}$ *and* $\{2, 3, 3\}$ *are good.*

Proof. This follows immediately from Lemma 3 of (**4**).

We now prove a lemma similar to this lemma but with one extra condition required.

LEMMA 1. *If G is a group, $AB = G$, A has four elements, and two elements of A have a common square, then A or B is periodic.*

Proof. Let A have four elements a, b, c, d with $a^2 = b^2$. Then

$$(a, b, c, d)B = G. \tag{1}$$

Multiplying (1) by a and by b we obtain

$$(a^2, ab, ac, ad)B = G, \tag{2}$$

$$(ba, b^2, bc, bd)B = G. \tag{3}$$

Using $a^2 = b^2$ and comparing (2) and (3) we see that

$$(ac, ad)B = (bc, bd)B.$$

Now, if acB and bcB have an element in common, then so also have aB and bB, which contradicts (1). Therefore

$$acB = bdB, \qquad adB = bcB.$$

It follows that either B is periodic or $ac = bd$ and $ad = bc$. In this case

$$\begin{aligned} ab^{-1}A = ab^{-1}(a, b, c, d) &= (a^2b^{-1}, a, b^{-1}ac, b^{-1}ad) \\ &= (b^2b^{-1}, a, b^{-1}bd, b^{-1}bc) = (b, a, d, c) = A. \end{aligned}$$

Therefore A is periodic.

THEOREM 5. *The groups G of types* $\{2^2, 2, 2\}$ *and* $\{2^2, 2, 2, 2\}$ *are good.*

Proof. It may be assumed that one factor has either two or four elements. In the first case one factor is periodic by Lemma 3 of (**4**). Since there are only two squares among the elements of G, it follows in the second case by Lemma 1 that one factor is periodic.

THEOREM 6. *The group G of type* $\{2^3, 2, 2\}$ *is good.*

Proof. Let $AB = G$. It may be assumed that A has two or four elements. If A has two elements, then A or B is periodic by Lemma 3 of (**4**). Let A have four elements. Then it can be assumed that no two elements of A have a common square.

Let a, b, c of orders 8, 2, 2 respectively generate G. Then (a, a^4b, c), (a, b, a^4c), and (a, a^4b, a^4c) are other systems of generators of G. Let

$$A = \sum a^{\alpha_i} b^{\beta_i} c^{\gamma_i} = \sum a^{\alpha_i+4\beta_i} (a^4b)^{\beta_i} c^{\gamma_i} = \sum a^{\alpha_i+4\gamma_i} (a^4c)^{\gamma_i} b^{\beta_i}$$
$$= \sum a^{\alpha_i+4\beta_i+4\gamma_i} (a^4b)^{\beta_i} (a^4c)^{\gamma_i},$$
$$B = \sum a^{\lambda_i} b^{\mu_i} c^{\nu_i} = \sum a^{\lambda_i+4\mu_i} (a^4b)^{\mu_i} c^{\nu_i} = \sum a^{\lambda_i+4\nu_i} (a^4c)^{\nu_i} b^{\mu_i}$$
$$= \sum a^{\lambda_i+4\mu_i+4\nu_i} (a^4b)^{\mu_i} (a^4c)^{\nu_i}.$$

No two exponents α_i in A can be congruent modulo 4. From $AB = G$ it follows that

$$(\sum x^{\alpha_i})(\sum x^{\lambda_i}) \equiv (\sum x^{\alpha_i+4\beta_i})(\sum x^{\lambda_i+4\mu_i}) \equiv (\sum x^{\alpha_i+4\gamma_i})(\sum x^{\lambda_i+4\nu_i})$$
$$\equiv (\sum x^{\alpha_i+4\beta_i+4\gamma_i})(\sum x^{\lambda_i+4\mu_i+4\nu_i}) \equiv 4(1+x+\ldots+x^7) \pmod{(x^8-1)}.$$

Therefore $F_8(x) = x^4+1$ divides (i) $\sum x^{\alpha_i}$ or $\sum x^{\lambda_i}$, (ii) $\sum x^{\alpha_i+4\beta_i}$ or $\sum x^{\lambda_i+4\mu_i}$, (iii) $\sum x^{\alpha_i+4\gamma_i}$ or $\sum x^{\lambda_i+4\nu_i}$, (iv) $\sum x^{\alpha_i+4\beta_i+4\gamma_i}$ or $\sum x^{\lambda_i+4\mu_i+4\nu_i}$. From the above assumption about the exponents α_i it follows that $F_8(x)$ divides $\sum x^{\lambda_i}$, $\sum x^{\lambda_i+4\mu_i}$, $\sum x^{\lambda_i+4\nu_i}$, $\sum x^{\lambda_i+4\mu_i+4\nu_i}$.

These results are now used to show that a^4 is a period of B. The following notation is used. We say that

$$(k_1, l_1, m_1) \equiv (k_2, l_2, m_2) \quad \text{modulo } (8, 2, 2)$$

if $\quad k_1 \equiv k_2 \pmod 8, \quad l_1 \equiv l_2 \pmod 2, \quad m_1 \equiv m_2 \pmod 2.$

It is shown that $(4, 0, 0)$ is an additive period of the three-tuples $(\lambda_i, \mu_i, \nu_i)$ modulo $(8, 2, 2)$. From the above results 4 is an additive period of each of the sets λ_i, $\lambda_i+4\mu_i$, $\lambda_i+4\nu_i$, $\lambda_i+4\mu_i+4\nu_i$ modulo 8. Since B can contain no element twice, any given number $\lambda_i = \lambda$ occurs at most four times.

If $\lambda_i = \lambda$ occurs four times, then the pairs (μ_i, ν_i) occurring with it must be $(0, 0)$, $(0, 1)$, $(1, 0)$, $(1, 1)$. But $\lambda_i \equiv \lambda+4 \pmod 8$ must also occur four times, and again the pairs (μ_i, ν_i) occurring with it must be $(0, 0)$, $(0, 1)$, $(1, 0)$, $(1, 1)$. Thus $(4, 0, 0)$ is a period of these sets modulo $(8, 2, 2)$. Let $\lambda_i = \lambda$ occur three times. Let the missing three-tuple be (λ, μ, ν). Then from these three-tuples $\lambda+4\mu$ occurs once and $\lambda+4\mu+4$ occurs twice among $\lambda_i+4\mu_i$. Similarly $\lambda+4\nu$ occurs once and $\lambda+4\nu+4$ occurs twice among $\lambda_i+4\nu_i$. Now $\lambda_i \equiv \lambda+4 \pmod 8$ must arise three times. It is the only other number λ_i giving rise to numbers congruent to $\lambda \pmod 4$ in any of the sets. Therefore from these $\lambda+4\mu$ must arise twice, $\lambda+4\mu+4$ must arise once in $\lambda_i+4\mu_i$, and $\lambda+4\nu$ must arise twice and $\lambda+4\nu+4$ must arise once in $\lambda_i+4\nu_i$. It is easily verified that $(\lambda+4, \mu, \nu)$ must again be the missing three-tuple. Thus $(4, 0, 0)$ is a period of these sets modulo $(8, 2, 2)$. Similarly, if $\lambda_i = \lambda$ occurs once

only, say as (λ, μ, ν), then again, using the periodicity of λ_i, $\lambda_i+4\mu_i$, and $\lambda_i+4\nu_i$, it can be shown that $(\lambda+4, \mu, \nu)$ occurs. Thus $(4, 0, 0)$ is again a period modulo $(8, 2, 2)$.

The remaining case is that in which $\lambda_i = \lambda$ occurs twice. In this case the corresponding three-tuples may be of the form

$$(\lambda, \mu, \nu),\ (\lambda, \mu+1, \nu); \quad (\lambda, \mu, \nu),\ (\lambda, \mu, \nu+1); \quad \text{or} \quad (\lambda, \mu, \nu),\ (\lambda, \mu+1, \nu+1).$$

The first two cases are similar and only the first and third are considered. In the first case $\lambda+4$ must occur twice, and from the periodicity of $\lambda_i+4\nu_i$ it follows that ν must occur twice with it. The corresponding μ_i must be distinct and so must be μ and $\mu+1$. Thus $(4, 0, 0)$ is a period of these sets modulo $(8, 2, 2)$. In the third case $\lambda+4$ must occur twice and μ and $\mu+1$ once each and ν and $\nu+1$ once each with it. This allows two possibilities:

$$(\lambda+4, \mu, \nu),\ (\lambda+4, \mu+1, \nu+1) \quad \text{or} \quad (\lambda+4, \mu, \lambda+1),\ (\lambda+4, \mu+1, \nu).$$

We have not yet used the fact that 4 is a period of $\lambda_i+4\mu_i+4\nu_i$ modulo 8. Numbers congruent to λ modulo 4, can only arise from $\lambda_i = \lambda$ or $\lambda_i \equiv \lambda+4 \pmod 8$. But the numbers

$$(\lambda, \mu, \nu), \quad (\lambda, \mu+1, \nu+1), \quad (\lambda+4, \mu, \nu+1), \quad (\lambda+4, \mu+1, \nu)$$

all give rise to $\lambda+4\mu+4\nu$ modulo 8. Therefore these sets cannot arise. Thus $(4, 0, 0)$ is again a period modulo $(8, 2, 2)$. It follows that a^4 is a period of B. This completes the proof.

THEOREM 7. *If G is a group of type $\{2^\lambda, 2\}$, then G is good.*

Proof. We may suppose that $\lambda > 1$ since the case $\lambda = 1$ has been dealt with by de Bruijn. Let $2^{\lambda-1} = m$. Let a and b, of orders $2m$ and 2 respectively, generate G. Then a and $a^m b$ also generate G. Let

$$A = \sum a^{\alpha_i} b^{\beta_i} = \sum a^{\alpha_i + m\beta_i} (a^m b)^{\beta_i},$$
$$B = \sum a^{\lambda_i} b^{\mu_i} = \sum a^{\lambda_i + m\mu_i} (a^m b)^{\mu_i}.$$

Then from $AB = G$ it follows that

$$\left(\sum x^{\alpha_i}\right)\left(\sum x^{\lambda_i}\right) \equiv \left(\sum x^{\alpha_i + m\beta_i}\right)\left(\sum x^{\lambda_i + m\mu_i}\right) \equiv 2(1+x+\ldots+x^{2m-1}) \pmod{(x^{2m}-1)}.$$

Therefore $F_{2m}(x) = (x^m+1)$ divides (i) $\sum x^{\alpha_i}$ or $\sum x^{\lambda_i}$, (ii) $\sum x^{\alpha_i+m\beta_i}$ or $\sum x^{\lambda_i+m\mu_i}$.

The two cases to consider are that in which $F_{2m}(x)$ divides two polynomials derived from the same factor and that in which $F_{2m}(x)$ divides one polynomial derived from each factor.

In the first case we may assume that $F_{2m}(x)$ divides $\sum x^{\alpha_i}$ and

$\sum x^{\alpha_i+m\beta_i}$. As in the previous theorem we use these results to show that a^m is a period of A, i.e. that $(m, 0)$ is an additive period of (α_i, β_i) modulo $(2m, 2)$. Now $\alpha_i = \alpha$ can occur at most twice. If $\alpha_i = \alpha$ occurs twice, then $\alpha_i \equiv \alpha+m \pmod{2m}$ also occurs twice, and in each case the numbers β_i must be different, and so must be 0 and 1. Thus $(m, 0)$ is a period of these sets modulo $(2m, 2)$. If $\alpha_i = \alpha$ occurs once only, say as (α, β), then $\alpha+m$ must also occur once only. In $\alpha_i+m\beta_i$, $\alpha+m\beta$ occurs once. Therefore $\alpha+m\beta+m$ must also occur once. This can arise only from $(\alpha+m, \beta)$. Thus $(m, 0)$ is a period of these sets modulo $(2m, 2)$. It follows that a^m is a period of A.

In the second case it may be assumed that $F_{2m}(x)$ divides $\sum x^{\alpha_i}$ and $\sum x^{\lambda_i+m\mu_i}$. Thus, if (α, β) occurs among (α_i, β_i), so also does either $(\alpha+m, \beta)$ or $(\alpha+m, \beta+1)$ modulo $(2m, 2)$. Also, if (λ, μ) occurs among (λ_i, μ_i), so also does $(\lambda, \mu+1)$ or $(\lambda+m, \mu)$ modulo $(2m, 2)$. If every time (λ, μ) occurs, so also does $(\lambda, \mu+1)$, then b is a period of B. Suppose that (λ, μ) and $(\lambda+m, \mu)$ occur. Let (α, β) occur in A. Then $(\alpha+m, \beta)$ cannot occur. For, if it occurred, then the element $a^{\alpha+\lambda+m}b^{\beta+\mu}$ would occur twice in AB as $(a^{\alpha}b^{\beta})(a^{\lambda+m}b^{\mu})$ and as $(a^{\alpha+m}b^{\beta})(a^{\lambda}b^{\mu})$. Therefore, if (α, β) occurs, $(\alpha+m, \beta+1)$ must also occur. It follows that $a^m b$ is a period of A. This completes the proof.

THEOREM 8. *The group G of type $\{2^2, 2^2\}$ is good.*

Proof. Let a and b, each of order 4, generate G. Let $AB = G$. If A has two elements, then, by Lemma 3 of (**4**), A or B is periodic. We may assume that A and B have each four elements. Then by Lemma 1 it may be assumed that no two elements of A and no two elements of B have a common square. There are only four squares in B, namely e, a^2, b^2, a^2b^2. It follows that the squares of the elements of A and of B take these once each. Therefore A and B are of the form

$$e, \quad (a \text{ or } a^3).(e \text{ or } b^2), \quad (e \text{ or } a^2).(b \text{ or } b^3), \quad (a \text{ or } a^3).(b \text{ or } b^3).$$

Let
$$A = \sum a^{\alpha_i}b^{\beta_i}, \qquad B = \sum a^{\lambda_i}b^{\mu_i}.$$
Then from $AB = G$ it follows that

$$\left(\sum x^{\alpha_i}\right)\left(\sum x^{\lambda_i}\right) \equiv 4(1+x+x^2+x^3) \pmod{(x^4-1)}.$$

Therefore $F_4(x)$ divides $\sum x^{\alpha_i}$ or $\sum x^{\lambda_i}$. It may be assumed that $F_4(x)$ divides $\sum x^{\alpha_i}$. Then the numbers α_i are 0, 1, 2, 3 or 0, 0, 2, 2. From the form of A given above they must be 0, 1, 2, 3. If $a^3 = c$, then $c^3 = a$ and, if $b^3 = d$, then $d^3 = b$. Thus by renaming generators, if necessary, it may be assumed that A is of the form

$$e, \quad a(e \text{ or } b^2), \quad a^2b, \quad a^3(b \text{ or } b^3).$$

Now, *if* g_1 and g_2 are different elements of A, then $g_1g_2^{-1}$ is not in B. For this would give g_1 occurring twice in AB as

$$g_1 = (g_1)(e) = (g_2)(g_1g_2^{-1}).$$

If A is e, a, a^2b, a^3b, then letting

$$g_1 = e, g_2 = a^3b;\ g_1 = a^3b, g_2 = e;\ g_1 = a, g_2 = a^2b;\ g_1 = a^2b, g_2 = a$$

we see that B can have no element whose square is a^2b^2. If A is e, ab^2 a^2b, a^3b, then letting

$$g_1 = e, g_2 = ab^2; g_1 = ab^2, g_2 = e; g_1 = a^2b, g_2 = a^3b; g_1 = a^3b, g_2 = a^2b$$

we see that B can have no element whose square is a^2. If A is e, a, a^2b, a^3b^3, then letting

$$g_1 = e, g_2 = a; g_1 = a, g_2 = e; g_1 = a^2b, g_2 = a^3b^3; g_1 = a^3b^3, g_2 = a^2b$$

we see that B can have no element whose square is a^2. Finally, if A is e, ab^2, a^2b, a^3b^3, then letting

$$g_1 = e, g_2 = a^3b^3;\quad g_1 = a^3b^3, g_2 = e;\quad g_1 = ab^2, g_2 = a^2b;$$
$$g_1 = a^2b, g_2 = ab^2$$

we see that B can have no element whose square is a^2b^2. It follows that no factorization exists in which A and B are both non-periodic. Therefore G is good.

4. Factorizations of infinite abelian groups

In this section extensions of previous results on finite abelian groups to certain infinite abelian groups are made. It is shown that, where a group of type $\{p^\lambda\}$ occurred in the finite case, it may be replaced by a group of type $\{p^\infty\}$. Every element of a group of type $\{p^\infty\}$ has finite order p^λ, where λ is some integer. Every proper subgroup is a finite cyclic group of order p^λ for some integer λ. If a and b are elements of orders p^λ and p^μ, where $\lambda > \mu$, then ab has order p^λ. If $\lambda = \mu$, then ab has order less than or equal to p^λ. The convention is adopted that, if λ is an integer, then $\lambda < \infty$.

THEOREM 9. *If G is a group of type $\{p^\infty\}$, where p is a prime, and $AB = G$, where the number of elements in A is finite, then A or B is periodic.*

Proof. Since every element of A has finite order, there exists an integer λ such that every element of A has order less than or equal to p^λ. For each integer μ let B_μ denote the set of elements of B with order less than or equal to p^μ. Let $B-B_\mu$ denote the remaining elements of B. If $\mu \geqslant \lambda$, $A(B-B_\mu)$ contains no element of order less than or

equal to p^μ but every element of AB_μ has order less than or equal to p^μ. Since $AB = G$, it follows that $AB_\mu = G_\mu$, where G_μ is the subgroup of G of order p^μ. By the result of Hajós (**3**), A or B_μ is periodic. Therefore, if A is not periodic, B_μ is periodic for all $\mu \geqslant \lambda$. Since any power of a period of B_μ is also a period, it follows that every element of G of order p is a period of B_μ. Let g be an element of order p. Let b be in B. Since b has finite order, there exists an integer $\mu \geqslant \lambda$ such that b is in B_μ. Therefore gb is in B_μ and so in B. It follows that g is a period of B. This completes the proof.

THEOREM 10. *If G is a direct product of a subgroup H of type $\{p^\infty\}$ and a subgroup K of type $\{q\}$, where p and q are distinct primes, $AB = G$, and the number of elements in A is finite, then A or B is periodic.*

Proof. Any element of G can be expressed uniquely as hk with h in H and k in K. Let $A = \sum h_i k_i$, where the elements h_i are in H. Since the number of elements in A is finite, there exists an integer λ such that every element h_i occurring in the expression for A has order less than or equal to p^λ. For each positive integer μ, let B_μ denote the set of elements b of B such that the greatest power of p dividing the order of b is less than or equal to p^μ. Let $B-B_\mu$ denote the remaining elements of B. For each $\mu \geqslant \lambda$, $A(B-B_\mu)$ contains no element whose order is not divisible by $p^{\mu+1}$, and AB_μ contains no element whose order is divisible by $p^{\mu+1}$. Since $AB = G = HK$, it follows that $AB_\mu = H_\mu K$, where H_μ is the cyclic subgroup of H of order p^μ. Then $H_\mu K$ is a group of type $\{p^\mu, q\}$ By de Bruijn's result (**2**), it follows that A or B_μ is periodic. If A is not periodic, then B_μ is periodic for every $\mu \geqslant \lambda$. Since any power of a period of B_μ is also a period of B_μ, it follows that either every element of order p or every element of order q is a period. One of these must be a period for infinitely many μ. Let g be a period of B_μ for infinitely many μ. Then for any $\mu \geqslant \lambda$ there exists $\nu \geqslant \mu$ such that g is a period of B_ν. Let b be any element of B. Then, since b has finite order, b is in some B_ν such that g is a period of B_ν. It follows that gb is in B_ν and so in B. Therefore g is a period of B. This completes the proof.

THEOREM 11. *If G is a direct product of a group H of type $\{2^\infty\}$ and a group K of type $\{2\}$ and $AB = G$, where A has a finite number of elements, then either A or B is periodic.*

Proof. Let $A = \sum h_i k_i$, where the elements h_i are in H. Since the number of elements in A is finite, there exists an integer λ such that

every element h_i occurring in the expression for A has order less than or equal to 2^λ. For each positive integer μ let B_μ denote the set of elements $b = hk$ of B, where h is in H and k is in K and such that the order of h is less than or equal to 2^μ. Then, as in the proof of the previous theorem, $AB_\mu = H_\mu K$ for all $\mu \geqslant \lambda$, where H_μ is the subgroup of H of type $\{2^\mu\}$. Then, by Theorem 7, either A or B_μ is periodic. If A is not periodic, B_μ is periodic for all $\mu \geqslant \lambda$. It follows that an element of order two is a period of B_μ in each case. Since G contains only three elements of order two, it follows that one of them must be a period of B_μ infinitely many times. As before it can be shown that this element is a period of B. This completes the proof.

THEOREM 12. *If G is the direct product of groups of type $\{p_i^{\lambda_i}\}$, where $i = 1, 2, \ldots, k$, the numbers p_i are distinct primes and the exponents λ_i are positive integers or infinity, $AB = G$, and the number of elements of A is a power of a prime, then either A or B is periodic.*

Proof. Let $G = H_1 . H_2 . \ldots . H_k$, where, for each i, H_i is a group of type $\{p_i^{\lambda_i}\}$. Then every element g of G can be expressed uniquely as $g = h_1 h_2 \ldots h_k$, where, for each i, h_i is in H_i. Let

$$A = \sum_j h_{1j} h_{2j} \ldots h_{kj},$$

where h_{ij} is in H_i for each i. Then, since the number of elements of A is finite, for each i there exists an integer ν_i such that every h_{ij} occurring in the expression for A has order less than or equal to $p_i^{\nu_i}$ and $\nu_i \leqslant \lambda_i$. Let

$$B_{\mu_1, \mu_2, \ldots, \mu_k}$$

be the set of elements of B such that $b = h_1 h_2 \ldots h_k$, where h_i is in H_i and has order less than or equal to $p_i^{\mu_i}$, where the integers μ_i are less than or equal to λ_i. Let $B - B_{\mu_1, \ldots, \mu_k}$ be the remaining elements of B.

Suppose that $\lambda_i \geqslant \mu_i \geqslant \nu_i$, where μ_i is an integer, for $1 \leqslant i \leqslant k$. Let H_{i,μ_i} denote the subgroup of H_i of order $p_i^{\mu_i}$. Then $A(B - B_{\mu_1, \ldots, \mu_k})$ contains no element of $H_{1,\mu_1} H_{2,\mu_2} \ldots H_{k,\mu_k}$, but every element of $AB_{\mu_1, \ldots, \mu_k}$ is in $H_{1,\mu_1} H_{2,\mu_2} \ldots H_{k,\mu_k}$. Since $AB = G$, it follows that

$$AB_{\mu_1, \ldots, \mu_k} = H_{1,\mu_1} \ldots H_{k,\mu_k}.$$

Since the numbers p_i are distinct primes, $H_{1,\mu_1} \ldots H_{k,\mu_k}$ is a finite cyclic group. Therefore, by Theorem 2 of (4), A or $B_{\mu_1, \ldots, \mu_k}$ is periodic.

If A is not periodic, then $B_{\mu_1, \ldots, \mu_k}$ is periodic for every set $\mu_1, \ldots, \mu_k$ such that $\lambda_i \geqslant \mu_i \geqslant \nu_i$. Let $\mu_{i,n} = n$ if $n \leqslant \lambda_i$ and $\mu_{i,n} = \lambda_i$ if $n > \lambda_i$ for each integer n. Then for all n greater than or equal to the maximum of $\nu_1, \ldots, \nu_k$, $B_{\mu_{1n}, \mu_{2n}, \ldots, \mu_{kn}}$ is periodic. Since any power of a period is

also a period, it follows that all elements of order p_1, all elements of order $p_2, \ldots$, or all elements of order p_k are periods of $B_{\mu_{1,n},\ldots,\mu_{k,n}}$. Therefore some element, say g, is a period for infinitely many n. Let b be in B. Since b has finite order, b is in $B_{\mu_{1,n},\ldots,\mu_{k,n}}$ for some n, and g is a period of this set. Therefore gb is in $B_{\mu_{1,n},\ldots,\mu_{k,n}}$ and so in B. It follows that g is a period of B. This completes the proof.

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ON THE SUM OF A PARTICULAR BILATERAL HYPERGEOMETRIC SERIES ${}_3H_3$

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1. It has been known for many years† that

$$ {}_2H_2\begin{bmatrix} a, b; \\ d, e \end{bmatrix} = \Gamma\begin{bmatrix} d, e, 1-a, 1-b, d+e-a-b-1; \\ d-a, d-b, e-a, e-b \end{bmatrix}, \tag{1.1} $$

where
$$ \Gamma\begin{bmatrix} a_1,..., a_n; \\ b_1,..., b_m \end{bmatrix} = \frac{\Gamma(a_1)...\Gamma(a_n)}{\Gamma(b_1)...\Gamma(b_m)}. $$

Recently H. S. Shukla (**5**) has obtained a formula equivalent to

$$ {}_3H_3\begin{bmatrix} 1+\tfrac{1}{2}\kappa, b, a; \\ \tfrac{1}{2}\kappa, 1+\kappa-b, w \end{bmatrix} $$
$$ = \frac{1+\kappa-a-w}{\kappa}\Gamma\begin{bmatrix} 1-a, 1-b, w, 1+\kappa-b, \kappa-a-2b+w-1; \\ w-a, w-b, \kappa-2b, 1+\kappa-a-b \end{bmatrix} \tag{1.2} $$

which, he remarks, gives the formula‡

$$ {}_3F_2\begin{bmatrix} a, 1+\tfrac{1}{2}a, b; \\ \tfrac{1}{2}a, w \end{bmatrix} = (w+b-a-1)\Gamma\begin{bmatrix} w, w-b-a-1; \\ w-a, w-b \end{bmatrix} \tag{1.3} $$

when $b = \kappa$ and κ, a are then replaced by a, b, and when $w = 1$ in (1.2) it reduces to‡

$$ {}_3F_2\begin{bmatrix} a, 1+\tfrac{1}{2}\kappa, b; \\ \tfrac{1}{2}\kappa, 1+\kappa-b \end{bmatrix} = \frac{\kappa-a}{\kappa}\Gamma\begin{bmatrix} 1+\kappa-b, \kappa-a-2b; \\ \kappa-2b, 1+\kappa-a-b \end{bmatrix}. \tag{1.4} $$

I give here a formula more general than (1.2), namely,

$$ {}_3H_3\begin{bmatrix} a, b, f+1; \\ d, e, f \end{bmatrix} = \lambda\Gamma\begin{bmatrix} d, e, 1-a, 1-b, d+e-a-b-2; \\ d-a, d-b, e-a, e-b \end{bmatrix}, \tag{1.5} $$

where
$$ \lambda = \{(f-a)(f-b)-(1+f-d)(1+f-e)\}/f. $$

This gives (1.1) when $f \to \infty$, and (1.2) when

$$ 1+2f = b+e. $$

† Dougall (**3**) § 13. For the notation used here, see Bailey (**2**).

‡ For (1.3) when b is a negative integer, see (**1**) § 4.5 (1.1), and for (1.4) see (**1**) § 6.4 (2).

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2. The proof of (1.5) is an easy deduction from (1.1).

Since
$$\frac{(f+1)_n}{(f)_n}=\frac{f+n}{f}=\frac{(e+n-1)+(f-e+1)}{f},$$
it follows that
$${}_3H_3\begin{bmatrix}a,b,f+1;\\ d,e,f\end{bmatrix}=\frac{1}{f}\left\{(e-1)_2H_2\begin{bmatrix}a,b;\\ d,e-1\end{bmatrix}+(f-e+1)_2H_2\begin{bmatrix}a,b;\\ d,e\end{bmatrix}\right\}$$
and the result follows from (1.1).

3. A more general result can be obtained. M. Jackson (4) has shown that
$${}_3H_3\begin{bmatrix}a,b,c;\\ d,e,f\end{bmatrix}=\Gamma\begin{bmatrix}d,e,f,1-a,1-b,c-b;\\ d-b,e-b,f-b,c,1+b-a\end{bmatrix}\times$$
$$\times{}_3F_2\begin{bmatrix}1+b-d,1+b-e,1+b-f;\\ 1+b-c,1+b-a\end{bmatrix}+$$
$$+(b\leftrightarrow c),\qquad(3.1)$$
where $(b\leftrightarrow c)$ means that the second term on the right is obtained from the first by interchange of b and c.

Now, when $c=f+n$, where n is a positive integer, the second expression on the right vanishes because of the factor $\Gamma(f-c)$ in the denominator, and we get
$${}_3H_3\begin{bmatrix}a,b,f+n;\\ d,e,f\end{bmatrix}=\frac{(f-b)_n}{(f)_n}\Gamma\begin{bmatrix}d,e,1-a,1-b;\\ d-b,e-b,1+b-a\end{bmatrix}\times$$
$$\times{}_3F_2\begin{bmatrix}1+b-d,1+b-e,1+b-f;\\ 1+b-f-n,1+b-a\end{bmatrix}.\qquad(3.2)$$

We now use the relation [(1) § 3.8 (1)] between $Fp(0;4,5)$, $Fn(5;0,3)$, and $Fn(3;0,5)$ with $c=f+n$, and we get
$${}_3F_2\begin{bmatrix}a,b,f+n;\\ e,f\end{bmatrix}=\Gamma\begin{bmatrix}e,e-a-b;\\ e-a,e-b\end{bmatrix}{}_3F_2\begin{bmatrix}a,b,-n;\\ a+b-e+1,f\end{bmatrix},$$
and this shows immediately that the series in (3.2) can be expressed in finite terms. In fact
$${}_3H_3\begin{bmatrix}a,b,f+n;\\ d,e,f\end{bmatrix}=\frac{(f-b)_n}{(f)_n}\Gamma\begin{bmatrix}d,e,1-a,1-b,d+e-a-b-1;\\ d-a,d-b,e-a,e-b\end{bmatrix}\times$$
$$\times{}_3F_2\begin{bmatrix}1+b-d,1+b-e,-n;\\ 2+a+b-d-e,1+b-f-n\end{bmatrix}.\qquad(3.3)$$

A result symmetrical in a and b, and in d and e, can be obtained from this by using the formula $Fp(0; 4, 5) = Fp(0; 1, 4)$. Then (3.3) gives

$$
{}_3H_3\begin{bmatrix} a, b, f+n; \\ d, e, f \end{bmatrix} = \frac{(f-a)_n(f-b)_n}{(f)_n}\Gamma\begin{bmatrix} d, e, 1-a, 1-b, d+e-a-b-n-1; \\ d-a, d-b, e-a, e-b \end{bmatrix} \times
$$

$$
\times {}_3F_2\begin{bmatrix} d-f-n, e-f-n, -n; \\ 1+a-f-n, 1+b-f-n \end{bmatrix}. \tag{3.4}
$$

Now reverse the series on the right of (3.4), and we get

$$
{}_3H_3\begin{bmatrix} a, b, f+n; \\ d, e, f \end{bmatrix} = \Gamma\begin{bmatrix} d, e, 1-a, 1-b, d+e-a-b-n-1; \\ d-a, d-b, e-a, e-b \end{bmatrix} \times
$$

$$
\times \frac{(-1)^n(1+f-d)_n(1+f-e)_n}{(f)_n}\, {}_3F_2\begin{bmatrix} f-a, f-b, -n; \\ 1+f-d, 1+f-e \end{bmatrix}. \tag{3.5}
$$

For the cases $n = 0, 1$, this reduces to (1.1), (1.5). When $f \to \infty$, (3.3) is the more suitable form of the result.

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THE MINIMUM MODULUS OF A POLYNOMIAL ON THE UNIT CIRCLE

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1. LET $P_n(z) = \sum_0^n a_\nu z^\nu$ be a polynomial of degree n with $|a_\nu| \leqslant 1$ $(0 \leqslant \nu \leqslant n)$. Then, if

$$m(P_n) = \min_{|z|=1} |P_n(z)|,$$

we get
$$m^2(P_n) \leqslant \frac{1}{2\pi} \int_0^{2\pi} |P_n(e^{i\theta})|^2 \, d\theta = \sum_0^n |a_\nu|^2. \tag{1}$$

From (1) it follows that

$$m(P_n) \leqslant (n+1)^{\frac{1}{2}}.$$

The following problems were posed by Dr. P. Erdős and conveyed to me by Professor W. K. Hayman. How large can $m(P_n)$ become when

(i) $|a_\nu| \leqslant 1 \quad (0 \leqslant \nu \leqslant n)$, (ii) $|a_\nu| = 1 \quad (0 \leqslant \nu \leqslant n)$,

(iii) $a_\nu = \pm 1 \quad (0 \leqslant \nu \leqslant n)$?

In this paper I shall give an answer to (i) by proving the theorem

THEOREM. *There is an absolute positive constant A such that for any integer n $(\geqslant 1)$ there is a polynomial*

$$Q_n(z) = \sum_0^n b_\nu^{(n)} z^\nu \quad (|b_\nu^{(n)}| \leqslant 1;\ 0 \leqslant \nu \leqslant n)$$

for which
$$m(Q_n) \geqslant An^{\frac{1}{2}}.$$

2. Two lemmas are required.

LEMMA 1. *Let*
$$f(z) = \sum_1^\infty e^{i\nu \log \nu} z^\nu = \sum_1^\infty c_\nu z^\nu.$$

For all $\rho < 1$ we have

$$|f(z)| < K(1-\rho)^{-\frac{1}{2}} \quad (|z| = \rho),$$

and, for a sequence $\rho_1 < \rho_2 < \ldots \to 1$ with $\rho_n \geqslant \rho_{n+1}^q$, q being a fixed positive integer,

$$|f(z)| > K^{-1}(1-\rho_n)^{-\frac{1}{2}} \quad (|z| = \rho_n),$$

where K is an absolute positive constant.

This result is known (**1**).

LEMMA 2. *If*

$$s_n(\rho,\theta) = \sum_1^n c_\nu(\rho e^{i\theta})^\nu \quad (n \geqslant 1), \qquad s_0(\rho,\theta) = 0$$

and
$$S_n(\rho,\theta) = \sum_1^n s_\nu(\rho,\theta) \quad (n \geqslant 1), \qquad S_0(\rho,\theta) = 0,$$

then
$$|S_n(\rho,\theta)| < Kn(1-\rho)^{-\frac{1}{2}} \quad (\rho < 1;\ |\theta| \leqslant \pi).$$

This follows from Lemma 1 and a known result [(**2**) 236].

It is known that $s_n(\rho,\theta) = O(n^{\frac{1}{2}})$ uniformly in n, ρ ($\leqslant 1$), and θ [(**3**) 118]. Use of this result would simplify the proof of the theorem to some extent, but, as it lies deeper than Lemma 1, I have avoided its use.

3. If $z = \rho_n e^{i\theta}$, then

$$\begin{aligned} f(z) &= \sum_1^\infty c_\nu(\rho_n^{\frac{1}{2}} e^{i\theta})^\nu \rho_n^{\frac{1}{2}\nu} \\ &= \sum_1^\infty \{s_\nu(\rho_n^{\frac{1}{2}},\theta) - s_{\nu-1}(\rho_n^{\frac{1}{2}},\theta)\}\rho_n^{\frac{1}{2}\nu} \\ &= (1-\rho_n^{\frac{1}{2}}) \sum_1^\infty s_\nu(\rho_n^{\frac{1}{2}},\theta)\rho_n^{\frac{1}{2}\nu}. \end{aligned}$$

Repeating this procedure we find that

$$f(z) = (1-\rho_n^{\frac{1}{2}})^2 \sum_1^\infty S_\nu(\rho_n^{\frac{1}{2}},\theta)\rho_n^{\frac{1}{2}\nu}. \tag{2}$$

Using Lemma 2 we have

$$\begin{aligned} \left|\sum_{N+1}^\infty S_\nu(\rho_n^{\frac{1}{2}},\theta)\rho_n^{\frac{1}{2}\nu}\right| &< K(1-\rho_n^{\frac{1}{2}})^{-\frac{1}{2}} \sum_{N+1}^\infty \nu\rho_n^{\frac{1}{2}\nu} \\ &= K(1-\rho_n^{\frac{1}{2}})^{-\frac{1}{2}}\left\{\frac{(N+1)\rho_n^{\frac{1}{2}(N+1)}}{(1-\rho_n^{\frac{1}{2}})} + \frac{\rho_n^{1+\frac{1}{2}N}}{(1-\rho_n^{\frac{1}{2}})^2}\right\} \\ &= K(1-\rho_n^{\frac{1}{2}})^{-\frac{3}{2}}\rho^{\frac{1}{2}(N+1)}\{N+1+\rho_n^{\frac{1}{2}}(1-\rho_n^{\frac{1}{2}})^{-1}\}. \end{aligned} \tag{3}$$

We now choose
$$N = \lambda_n = [\alpha(1-\rho_n^{\frac{1}{2}})^{-1}],$$

where α is a positive constant to be determined later. Hence

$$\begin{aligned} \rho_n^{\frac{1}{2}(N+1)} &= \exp\{(N+1)\log(1-(1-\rho_n^{\frac{1}{2}}))\} \\ &< \exp\{-(N+1)(1-\rho_n^{\frac{1}{2}})\} \\ &< e^{-\alpha}. \end{aligned}$$

From (3) and the above we find that

$$\Big|\sum_{N+1}^{\infty} S_\nu(\rho_n^{\frac{1}{2}},\theta)\rho_n^{\frac{1}{2}\nu}\Big| < K\Big(\frac{N+1}{\alpha}\Big)^{\frac{3}{2}} e^{-\alpha}\Big(N+1+\frac{N+1}{\alpha}\Big)$$
$$= K\alpha^{-\frac{5}{2}}(1+\alpha)e^{-\alpha}(N+1)^{\frac{5}{2}}$$
$$\leqslant K2^{\frac{5}{2}}\alpha^{-\frac{5}{2}}(1+\alpha)e^{-\alpha}N^{\frac{5}{2}}. \qquad (4)$$

Therefore, from Lemma 1, (2), (4), and the definition of $N = \lambda_n$,

$$\Big|\sum_{1}^{N} S_\nu(\rho_n^{\frac{1}{2}},\theta)\rho^{\frac{1}{2}\nu}\Big| > K^{-1}2^{-1}\alpha^{-\frac{5}{2}}N^{\frac{5}{2}} - K2^{\frac{5}{2}}\alpha^{-\frac{5}{2}}(1+\alpha)e^{-\alpha}N^{\frac{5}{2}}$$
$$= \alpha^{-\frac{5}{2}}\{K^{-1}2^{-1} - K2^{\frac{5}{2}}(1+\alpha)e^{-\alpha}\}N^{\frac{5}{2}} \qquad (5)$$

for $|\theta| \leqslant \pi$.

Obviously
$$\sum_{1}^{N} S_\nu(\rho_n^{\frac{1}{2}},\theta)\rho_n^{\frac{1}{2}\nu}$$
is a polynomial of degree N in $e^{i\theta}$, which we denote by $\sum_{0}^{N} d_\nu e^{i\nu\theta}$. Now, with $0 \leqslant \nu \leqslant N$,

$$d_\nu = c_\nu \rho_n^{\frac{1}{2}\nu}\{\rho_n^{\frac{1}{2}\nu} + 2\rho_n^{\frac{1}{2}(\nu+1)} + \ldots + (N+1-\nu)\rho_n^{\frac{1}{2}N}\},$$

and so

$$|d_\nu| < \rho_n^{\frac{1}{2}\nu}\{\rho_n^{\frac{1}{2}\nu} + \ldots + (N+1-\nu)\rho_n^{\frac{1}{2}N}\}$$
$$< (1-\rho_n^{\frac{1}{2}})^{-2} < \Big(\frac{N+1}{\alpha}\Big)^2 < 4\alpha^{-2}N^2. \qquad (6)$$

Let $e_\nu = (\alpha^2/4N^2)\,d_\nu$ $(0 \leqslant \nu \leqslant N)$, and we get, from (5) and (6),

$$\Big|\sum_{0}^{N} e_\nu z^\nu\Big| > \tfrac{1}{4}N^{\frac{1}{2}}\alpha^{-\frac{1}{2}}\{K^{-1}2^{-1} - K2^{\frac{5}{2}}(1+\alpha)e^{-\alpha}\} \quad (|z| = 1). \qquad (7)$$

Choosing α large enough and then keeping it fixed we see, from (7), that there is an absolute constant A_1 such that

$$\Big|\sum_{0}^{N} e_\nu z^\nu\Big| > A_1 N^{\frac{1}{2}} \quad (|z| = 1),$$

for the sequence $N = \lambda_n$.

For the sequence $N = \lambda_n$ we define

$$Q_N(z) = \sum_{0}^{N} e_\nu z^\nu. \qquad (8)$$

It follows from Lemma 1 that

$$\lambda_{n+1} = [\alpha(1-\rho_{n+1}^{\frac{1}{2}})^{-1}] < [\alpha(1-\rho_n^{1/(2q)})^{-1}]$$
$$< [\alpha q(1-\rho_n^{\frac{1}{2}})^{-1}] < q(\lambda_n+1) \leqslant 2q\lambda_n.$$

Hence, if, for $\lambda_n \leqslant \nu < \lambda_{n+1}$, we define

$$Q_\nu(z) = Q_{\lambda_n}(z), \qquad (9)$$

then we get, for $|z| = 1$,

$$|Q_\nu(z)| > A_1 \lambda_n^{\frac{1}{2}} > A_1(2q)^{-\frac{1}{2}}\nu^{\frac{1}{2}}.$$

Finally, for $1 \leqslant \nu < \lambda_1$, we define

$$Q_\nu(z) = 1. \tag{10}$$

The theorem follows from (8), (9), (10) with

$$A = \min\{\lambda_1^{-\frac{1}{2}}, A_1(2q)^{-\frac{1}{2}}\}.$$

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ON AN EXPANSION IN EXPONENTIAL SERIES (II)

By S. VERBLUNSKY (*Belfast*)

[Received 1 January 1958]

1. THE notation of the introduction of the first paper (1) will be used without further explanation. We begin with the analogue of the Riesz–Fischer theorem for the expansion we are considering. We have the theorem:

THEOREM 1. *If a_ν $(\nu = 1, 2,...)$ is a sequence of complex numbers such that $\sum |a_\nu|^2 < \infty$, then there is a function $f \in L_2(0, 1)$ such that*

$$a_\nu = \frac{1}{A'(\lambda_\nu)} \int_0^1 f(t)\phi_\nu(t)\, dt \quad (\nu = 1, 2,...).$$

The converse is not true without some further restriction on the function $A(z)$. I state without proof that it is possible to define $k(u)$ and a function f which is c.b.v. (continuous and of bounded variation), so that $\limsup |a_\nu| = \infty$. The function $A(z)$ which corresponds to $k(u)$ in this example has the property that pairs of its zeros are arbitrarily near to one another. When this does not happen, the converse of Theorem 1 holds. We have the theorem:

THEOREM 2. *If the set of distances between the zeros of $A(z)$ has a positive lower bound, and if $f \in L_2(0, 1)$, then $\sum |a_\nu|^2 < \infty$, where the a_ν are defined by* (1).

We consider finally the solution of the integral equation

$$\int_0^1 k(u)f(x+u)\, du = g(x), \tag{2}$$

where g is given for all x, f is given in $(0, 1)$, and it is required to define f outside this interval so as to satisfy (2) for all x. It is not difficult to find necessary and sufficient conditions that there shall be a unique solution belonging to a suitably chosen class of functions. I shall not do this here; such conditions have been stated without proof by Lyubič [(2) 194] in the case where $k(u)$ has bounded derivatives. We wish to obtain the solution of (2) in the form of a series $d_0 + \sum d_\nu e^{\lambda_\nu x}$, where the d's depend only on the given g and on the values of f in $(0, 1)$.

One must compromise between generality in the enunciation and simplicity in the proof. We shall consider the case in which the solution $f(x)$ is c.b.v. in every finite interval. The relevant existence-and-uniqueness theorem is as follows. *If* (2) *has a solution $f(x)$ which is c.b.v. in every finite interval, then $g'(x)$ has the same property, and the initial conditions*

$$\int_0^1 k(u)f(u)\,du = g(0), \qquad \int_0^1 k(u)\,df(u) = g'(0)$$

are satisfied. Conversely, if $g'(x)$ is c.b.v. in every finite interval and f is assigned and c.b.v. in $(0,1)$ *and satisfies the initial conditions, then the definition of f can be uniquely extended so as to be c.b.v. in every finite interval and to satisfy* (2) *for all x.*

Before enunciating the theorem, we remark that the notation of (**1**) (6) has the disadvantage that (**1**) (9) does not appear to be the case $\nu = 0$ of (**1**) (10). If we write $\lambda_0 = 0$ and

$$\omega_\nu(u) = k(u)+\lambda_\nu e^{-\lambda_\nu u}\int_u^1 k(v)e^{\lambda_\nu v}\,dv \quad (\nu = 0, 1,\ldots),$$

then (**1**) (8) becomes

$$\int_0^1 \omega_\nu \psi_\nu\,du = B'(\lambda_\nu) \quad (\nu = 0, 1,\ldots),$$

and (9), (10) of (**1**) can be written

$$\alpha_\nu = \frac{1}{B'(\lambda_\nu)}\int_0^1 f(u)\omega_\nu(u)\,du \quad (\nu = 0, 1,\ldots).$$

We now define
$$\beta_\nu(t) = \frac{1}{B'(\lambda_\nu)}\int_0^t g'(u)e^{-\lambda_\nu u}\,du$$

and
$$c_\nu = c_\nu(t) = \alpha_\nu+\beta_\nu(t) \quad (\nu = 0, 1,\ldots).$$

We have the theorem:

THEOREM 3. *Let $g'(x)$ be c.b.v. in every finite interval. Let $f(x)$ be a solution of* (2) *which is c.b.v. in every finite interval. Then*

$$\sum_0^{\nu_p} c_\nu e^{\lambda_\nu t}$$

converges, as $p \to \infty$, to $f(t)+g'(t)/2k(0)$, uniformly in every finite interval.

We establish this result by the method used for Theorem 1 of (**3**). It will therefore be sufficient to give a sketch of the proof, retaining

the details which are novel, and replacing the others by appropriate references to (3). This is done in § 6.

2. There is a positive integer κ such that the number of zeros of $A(z)$ in any ring $\rho \leqslant |z| \leqslant \rho+1$ is less than κ. Hence, giving ρ the value $|\lambda_\nu|$, we see that

$$|\lambda_{\kappa+\nu}| > |\lambda_\nu|+1 \quad (\nu = 1, 2,\ldots). \tag{3}$$

LEMMA 1. *If $|n-m| > \kappa$, then $|\lambda_n-\lambda_m| > |n-m|/2\kappa$.*

Proof. We may suppose that $n > m$. Let p be the positive integer which satisfies the inequalities

$$m+p\kappa \leqslant n < m+(p+1)\kappa.$$

Then $n-m < (p+1)\kappa$, and

$$p \geqslant \tfrac{1}{2}(p+1) > (n-m)/2\kappa. \tag{4}$$

Now
$$|\lambda_n-\lambda_m| \geqslant |\lambda_n|-|\lambda_m| \geqslant |\lambda_{m+p\kappa}|-|\lambda_m|$$
$$\geqslant \sum_{r=0}^{p-1} |\lambda_{m+(r+1)\kappa}|-|\lambda_{m+r\kappa}| > p$$

by (3). By (4), the result follows.

Write $\lambda_\nu = \xi_\nu+i\eta_\nu$. We recall that there is a positive constant C, which we may suppose to be greater than 1, such that $|\xi_\nu| < C$ for $\nu = 1, 2,\ldots$. Let

$$K = 12\kappa C, \tag{5}$$

so that $K > \kappa$.

LEMMA 2. *If $|n-m| > K$, then $|\lambda_n+\bar{\lambda}_m| > |n-m|/3\kappa$.*

Proof. We may suppose that $n > m$. By Lemma 1,

$$|\xi_n-\xi_m+i(\eta_n-\eta_m)| > (n-m)/2\kappa.$$

Hence

$$|\lambda_n+\bar{\lambda}_m| \geqslant |\eta_n-\eta_m| > (n-m)/2\kappa-2C > (n-m)/3\kappa.$$

LEMMA 3. *There is a positive constant A such that, if $\{a_n\}$ $(n = 1,\ldots, N)$ denotes any finite set of numbers, and*

$$\Phi(x) = \sum a_n e^{\lambda_n x}, \qquad \Psi(x) = \sum a_n e^{-\lambda_n x},$$

then

$$\int_0^1 |\Phi(x)|^2\, dx \leqslant A \sum |a_n|^2, \tag{6}$$

$$\int_0^1 |\Psi(x)|^2\, dx \leqslant A \sum |a_n|^2. \tag{7}$$

Proof. The proof is based on a study of (4), (5). We shall use P to denote a positive constant independent of the a_n, possibly different at

different occurrences. We write $Q = P\sum |a_n|^2$. Since $\Phi(-x) = \Psi(x)$, it suffices to prove that

$$\int_{-1}^{1} |\Phi(x)|^2\, dx \leqslant Q.$$

We have, for $0 < h < 2$,

$$\int_{-h}^{h} |\Phi|^2\, dx = \int_{-h}^{h} \sum a_n e^{\lambda_n x} \sum \bar{a}_m e^{\bar{\lambda}_m x}\, dx$$

$$= \sum |a_n|^2 \int_{-h}^{h} e^{2\xi_n x}\, dx + \sum' a_n \bar{a}_m \int_{-h}^{h} e^{(\lambda_n+\bar{\lambda}_m)x}\, dx$$

$$= J_1 + J_2,$$

where the accent denotes a summation subject to $n \neq m$. Then

$$0 < J_1 \leqslant \sum |a_n|^2 \int_{-2}^{2} e^{2\xi_n x}\, dx \leqslant Q.$$

Write $J_2 = \Sigma_1' + \Sigma_2'$, where in Σ_1' we have $|n-m| > K$ and in Σ_2' we have $0 < |n-m| \leqslant K$. Then

$$\Sigma_1' = 2\Sigma_1' \frac{a_n \bar{a}_m}{\lambda_n + \bar{\lambda}_m} \sinh(\lambda_n + \bar{\lambda}_m)h,$$

$$\int_{1}^{2} \Sigma_1'\, dh = 2\Sigma_1' \frac{a_n \bar{a}_m}{(\lambda_n + \bar{\lambda}_m)^2} \{\cosh 2(\lambda_n + \bar{\lambda}_m) - \cosh(\lambda_n + \bar{\lambda}_m)\}.$$

Hence
$$\left| \int_{1}^{2} \Sigma_1'\, dh \right| \leqslant P\Sigma_1' \frac{|a_n|\,|a_m|}{(n-m)^2},$$

by Lemma 2. Recalling that

$$2|a_n|\,|a_m| \leqslant |a_n|^2 + |a_m|^2,$$

we see that the last sum does not exceed

$$\sum_n |a_n|^2 \sum_m \frac{1}{(n-m)^2},$$

where, in the inner sum, $|n-m| > K$. Hence

$$\left| \int_{1}^{2} \Sigma_1'\, dh \right| \leqslant Q.$$

Further
$$|\Sigma_2'| \leqslant P\Sigma_2' |a_n|\,|a_m|.$$

This double sum consists of $2K$ expressions of the form $\sum |a_n||a_{n+\nu}|$, where ν has a fixed one of the values $\pm 1, \ldots, \pm K$. Hence

$$|\Sigma_2'| \leqslant Q, \qquad \left|\int_1^2 J_2\,dh\right| \leqslant Q.$$

Thus $$\int_1^2 dh \int_{-h}^{h} |\Phi|^2\,dx = \int_1^2 J_1\,dh + \int_1^2 J_2\,dh \leqslant Q.$$

Hence there is a ξ $(1 \leqslant \xi \leqslant 2)$ such that

$$\int_{-\xi}^{\xi} |\Phi|^2\,dx \leqslant Q.$$

Since $$\int_{-1}^{1} |\Phi|^2\,dx \leqslant \int_{-\xi}^{\xi} |\Phi|^2\,dx,$$

the lemma follows.

3. Proof of Theorem 1

Let $$F_m(x) = \sum_1^m a_n e^{\lambda_n x}.$$

If $\mu > m$, then

$$\int_0^1 |F_\mu - F_m|^2\,dx = \int_0^1 \left|\sum_{m+1}^{\mu} a_n e^{\lambda_n x}\right|^2 dx$$

$$\leqslant A \sum_{m+1}^{\mu} |a_n|^2,$$

by Lemma 3. Hence the sequence $\{F_m\}$ converges strongly in $L_2(0,1)$ to a function f. Further, for $\nu = 1, 2, \ldots$,

$$\int_0^1 f\phi_\nu\,dx = \lim_{m\to\infty} \int_0^1 F_m \phi_\nu\,dx$$

$$= a_\nu \int_0^1 e^{\lambda_\nu x}\phi_\nu(x)\,dx$$

$$= a_\nu A'(\lambda_\nu).$$

Hence the result.

4. Before proving Theorem 2, we show that the restriction on the zeros of $A(z)$ is equivalent to the condition that the numbers $|B'(\lambda_n)|$ exceed a positive constant.

LEMMA 4. *If there is a positive constant P such that*

$$|B'(\lambda_n)| > P \quad (n = 1, 2, \ldots),$$

then there is a positive d such that $|\lambda_n - \lambda_p| \geqslant d$ for $n \neq p$.

Proof. If not, there is a sub-sequence

$$\lambda_{n_\nu} = \xi_{n_\nu} + i\eta_{n_\nu} \tag{8}$$

such that $|\eta_{n_\nu}| \to \infty$, and such that to each ν corresponds a p_ν with the property, $\lambda_{n_\nu} - \lambda_{p_\nu} \to 0$. We may suppose that $\eta_{n_\nu} \to \infty$ or $-\infty$, say the former. Let R_m $(m = 0, 1, \ldots)$ denote the rectangle

$$|x| \leqslant C, \qquad 2m\pi \leqslant y \leqslant (2m+2)\pi.$$

To each ν there corresponds an m_ν such that $\lambda_{n_\nu} \in R_{m_\nu}$. Write

$$z_\nu = \lambda_{n_\nu} - 2\pi i m_\nu, \qquad z'_\nu = \lambda_{p_\nu} - 2\pi i m_\nu.$$

We may suppose that $\{z_\nu\}$ is convergent, say $z_\nu \to z_0$. Let

$$F_m(z) = B(z + 2\pi i m), \qquad G_\nu(z) = F_{m_\nu}(z).$$

Then
$$G_\nu(z_\nu) = G_\nu(z'_\nu) = 0. \tag{9}$$

The functions $G_\nu(z)$ being uniformly bounded in the rectangle

$$|x| \leqslant C+1, \qquad -1 \leqslant y \leqslant 2\pi+1,$$

there is a sub-sequence $\{G_{\nu_r}(z)\}$ and a function $G(z)$ such that

$$G_{\nu_r}(z) \to G(z), \qquad G'_{\nu_r}(z) \to G'(z),$$

uniformly in the rectangle

$$R'\colon\ |x| \leqslant C + \tfrac{1}{2}, \qquad -\tfrac{1}{2} \leqslant y \leqslant 2\pi + \tfrac{1}{2}.$$

Now $|G'_{\nu_r}(z_{\nu_r})| > P$. Hence $|G'(z_0)| \geqslant P$. Thus z_0 is a simple zero of G, and therefore there is a neighbourhood of z_0 in which G_{ν_r} has but one zero if r is sufficiently large. This contradicts (9).

LEMMA 5. *If there is a positive d such that $|\lambda_n - \lambda_p| \geqslant d$ for $n \neq p$, then there is a positive P such that $|B'(\lambda_n)| > P$ for $n = 1, 2, \ldots$.*

Proof. If not, there is a sub-sequence (8) such that $B'(\lambda_{n_\nu}) \to 0$. As before, we may suppose that $\eta_{n_\nu} \to \infty$. We define z_ν as above, and we may again suppose that $z_\nu \to z_0$. There is a sequence $\{G_{\nu_r}(z)\}$ converging uniformly to $G(z)$ in the rectangle R'. This time, we must prove that $G(z) \not\equiv 0$. This follows from the formula [see (1) (12)]

$$B^{(n)}(z) = k(1)e^z - \int_0^1 u^n e^{zu}\, dk(u).$$

This implies
$$G_\nu^{(n)}(0) = B^{(n)}(2\pi i m_\nu) = k(1) + \epsilon_n,$$

where $\epsilon_n \to 0$ uniformly in ν. Thus, for sufficiently large n, $G^{(n)}(0) \neq 0$.

The uniform convergence of $\{G_{\nu_r}\}$ together with $B'(\lambda_{n_\nu}) \to 0$ implies that $G(z_0) = 0$, $G'(z_0) = 0$. Since $G \not\equiv 0$, the disk $|z-z_0| \leqslant \frac{1}{3}d$ contains at least two zeros of G_{ν_r} if r is sufficiently large. But the zeros of $B(z)$ and those of G_{ν_r} are all simple. This contradicts the definition of d.

5. Proof of Theorem 2

We do not alter the value of a_ν by adding a constant to f. We may therefore suppose that

$$\int_0^1 f(u)k(u)\,du = 0. \tag{10}$$

Then
$$a_\nu A'(\lambda_\nu) = \int_0^1 f(t)e^{-\lambda_\nu t}\,dt \int_t^1 k(u)e^{\lambda_\nu u}\,du.$$

Writing

$$\lambda_\nu \int_t^1 k(u)e^{\lambda_\nu u}\,du = k(1)e^{\lambda_\nu} - k(t)e^{\lambda_\nu t} - \int_t^1 e^{\lambda_\nu u}\,dk(u),$$

we see from (10) that

$$a_\nu B'(\lambda_\nu) = k(1)e^{\lambda_\nu} r_\nu - s_\nu,$$

where
$$r_\nu = \int_0^1 f(t)e^{-\lambda_\nu t}\,dt, \qquad s_\nu = \int_0^1 dk(u) \int_0^u f(t)e^{\lambda_\nu (u-t)}\,dt.$$

By Lemma 5, it suffices to prove that

$$\sum |r_\nu|^2 < \infty \tag{11}$$

and

$$\sum |s_\nu|^2 < \infty. \tag{12}$$

To prove (11), let q_ν $(\nu = 1,\ldots, N)$ be any N numbers. Then

$$\sum r_\nu q_\nu = \int_0^1 f(t) \sum q_\nu e^{-\lambda_\nu t}\,dt,$$

$$\left|\sum r_\nu q_\nu\right|^2 \leqslant \int_0^1 |f|^2\,dt \int_0^1 \left|\sum q_\nu e^{-\lambda_\nu t}\right|^2 dt$$

$$\leqslant A \int_0^1 |f|^2\,dt \sum |q_\nu|^2,$$

by Lemma 3. This implies (11).

Further
$$\sum s_\nu q_\nu = \int_0^1 dk(u) \int_0^u f(t) \sum q_\nu e^{\lambda_\nu (u-t)}\,dt.$$

Let
$$\kappa(u) = \int_0^u |dk(u)|.$$

Then
$$|\sum s_\nu q_\nu| \leqslant \int_0^1 d\kappa(u)| \int_0^u f(t) S\, dt|,$$

where
$$S = \sum q_\nu e^{\lambda_\nu(u-t)}.$$

Hence
$$|\sum s_\nu q_\nu|^2 \leqslant \int_0^1 d\kappa(u) \int_0^1 \left|\int_0^u fS\, dt\right|^2 d\kappa(u).$$

But
$$\left|\int_0^u fS\, dt\right|^2 \leqslant \left(\int_0^1 |fS|\, dt\right)^2$$
$$\leqslant \int_0^1 |f|^2\, dt \int_0^1 |S|^2\, dt$$
$$\leqslant A \int_0^1 |f|^2\, dt \sum |q_\nu e^{\lambda_\nu u}|^2,$$

by Lemma 3; and
$$|e^{\lambda_\nu u}|^2 \leqslant e^{2C}.$$

Hence
$$|\sum s_\nu q_\nu|^2 \leqslant B \sum |q_\nu|^2,$$

where B is independent of the q_ν. This implies (12).

6. Proof of Theorem 3

We recall (17), (18) of (**1**). By subtracting appropriate constants from f and g, we may suppose that

$$g(0) = \int_0^1 k(u) f(u)\, du = 0. \tag{13}$$

If we adopt the method of (**3**) § 3, we obtain, for any a, b,

$$I_p = \frac{1}{\pi} \int_a^b f(x) \frac{\sin r_p(t-x)}{t-x}\, dx$$
$$= \frac{1}{2\pi i} \int_{q_p} \frac{e^{zt}}{B(z)} \Phi_b(z)\, dz - \frac{1}{2\pi i} \int_{q_p} \frac{e^{zt}}{B(z)} \Phi_a(z)\, dz, \tag{14}$$

where

$$\Phi_v(z) = z \int_0^v g(x) e^{-zx}\, dx - z \int_0^1 k(u) e^{zu}\, du \int_v^{v+u} f(x) e^{-zx}\, dx, \tag{15}$$

and q_p is the portion of the imaginary axis joining $-ir_p$ to ir_p, with small segments replaced by semicircles, if necessary, to avoid zeros of $B(z)$.

It is easy to prove that, for each ν, $\Phi_v(\lambda_\nu)$ is constant as a function of v. Hence

$$\Phi_v(\lambda_\nu) = \Phi_0(\lambda_\nu) = -\int_0^1 f(x)\{\omega_\nu(x)-k(x)\}\,dx$$

$$= -\int_0^1 f(x)\omega_\nu(x)\,dx,$$

by (13). If we now use the argument of (3) § 4, we find that

$$I_p = \sum_0^{\nu_p} \alpha_\nu e^{\lambda_\nu t}+J_p+K_p,$$

where
$$J_p = \frac{1}{2\pi i}\int_{C_p^+} \frac{e^{zt}}{B(z)}\Phi_b(z)\,dz, \tag{16}$$

$$K_p = \frac{1}{2\pi i}\int_{C_p^-} \frac{e^{zt}}{B(z)}\Phi_a(z)\,dz. \tag{17}$$

Let (α,β) be a closed interval contained in the open interval (a,b). Let t vary in (α,β). In what follows, the constant implied by an O-estimate is independent of t, and the convergence implied by an o-estimate is uniform in t. We shall prove that

$$J_p = \frac{1}{2\pi i}\int_{C_p^+} \frac{e^{zt}}{B(z)}\,dz\int_0^t e^{-zu}g'(u)\,du+o\,(1), \tag{18}$$

$$K_p = \frac{1}{2\pi i}\int_{C_p^-} \frac{e^{zt}}{B(z)}\,dz\int_0^t e^{-zu}g'(u)\,du-\frac{g'(t)}{2k(0)}+o\,(1). \tag{19}$$

It will follow that

$$J_p+K_p = \sum_0^{\nu_p} \beta_\nu(t)e^{\lambda_\nu(t)}-\frac{g'(t)}{2k(0)}+o\,(1).$$

Since $I_p = f(t)+o\,(1)$, this will prove the theorem.

In the expression (15) for $\Phi_v(z)$, the first term equals

$$-g(v)e^{-zv}+\int_0^v e^{-zx}g'(x)\,dx,$$

and the second term equals

$$\int_0^1 k(u)e^{zu}\,du\int_0^u f(x+v)\,d(-e^{-z(x+v)})$$

$$= f(v)e^{-zv}A(z)-g(v)e^{-zv}+e^{-zv}\int_0^1 df(x+v)\int_x^1 k(u)e^{z(u-x)}\,du.$$

Hence (18) will follow if we show that

$$\int\limits_{C_p^+} \frac{e^{zt}}{B(z)}\,dz\left\{\int\limits_t^b e^{-zx}g'(x)\,dx-f(b)e^{-zb}A(z)-\right.$$

$$\left.-e^{-zb}\int\limits_0^1 df(x+b)\int\limits_x^1 k(u)e^{z(u-x)}\,du\right\}$$

is $o(1)$. Of the three terms, consider the last, namely

$$\int\limits_{C_p^+} e^{z(t-b)}\frac{e^z}{B(z)}H\,dz,$$

where
$$H=\int\limits_0^1 df(x+b)\int\limits_x^1 k(u)e^{z(u-x-1)}\,du.$$

Since $k(u)$ is b.v., and $\operatorname{re} z(u-x-1)\leqslant 0$, we have $zH=O(1)$. Thus, the third term is $o(1)$. The consideration of the first and second terms is simpler.

Finally, (19) will follow if we show that

$$\int\limits_{C_p^-} \frac{e^{zt}}{B(z)}\,dz\left\{-\int\limits_a^t e^{-zx}g'(x)\,dx-f(a)e^{-za}A(z)-\right.$$

$$\left.-e^{-za}\int\limits_0^1 df(x+a)\int\limits_x^1 k(u)e^{z(u-x)}\,du\right\}=-\pi i g'(t)/k(0)+o(1).$$

The second and third terms are easily seen to be $o(1)$. As regards the first term, we observe that $1/B(z)$ is $O(1)$ on C_p^- and, for each $\arg z$ between $\frac{1}{2}\pi$ and $\frac{3}{2}\pi$, tends to $-1/k(0)$. Since $\operatorname{re} z(t-x)\leqslant 0$, we have

$$z\int\limits_a^t e^{z(t-x)}g'(x)\,dx=O(1),$$

so that, by Lebesgue's convergence theorem, the first term is

$$\frac{1}{k(0)}\int\limits_a^t g'(x)\,dx\int\limits_{C_p^-} e^{z(t-x)}\,dz+o(1)$$

$$=-\frac{2i}{k(0)}\int\limits_a^t g'(x)\frac{\sin r_p(t-x)}{t-x}\,dx+o(1)=-\pi i g'(t)/k(0)+o(1).$$

This completes the proof.

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ON THE ANALOGIES BETWEEN SOME SERIES CONTAINING BESSEL FUNCTIONS AND CERTAIN SPECIAL CASES OF THE WEBER–SCHAFHEITLIN INTEGRAL

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Two results of fundamental importance in a recent solution (**1**) of some dual integral equations are special cases of the Weber–Schafheitlin integral. The results in question, which are easily derived from the more general ones given by Watson (**2**), are

(i) *if m is zero or a positive integer, $\nu > -1-m$, and $k > 0$, then*

$$\left.\begin{aligned} I_1(\nu,m,k,r) &\equiv \int\limits_0^\infty x^{1-k}J_{\nu+2m+k}(x)J_\nu(rx)\,dx \\ &= \begin{cases} \dfrac{\Gamma(\nu+m+1)}{2^{k-1}\Gamma(\nu+1)\Gamma(m+k)}r^\nu(1-r^2)^{k-1}\mathscr{F}_m(k+\nu,\nu+1,r^2) & (0<r<1) \\ 0 \quad (1<r) \end{cases} \end{aligned}\right\}, \tag{1}$$

$\mathscr{F}_m(k+\nu,\nu+1,r^2)$ being Jacobi's polynomial (**3**);

(ii) *if m, n are zero or positive integers and $\nu > -m-n-k$, then*

$$\left.\begin{aligned} I_2(\nu,m,n,k) &\equiv \int\limits_0^\infty x^{-1}J_{\nu+2m+k}(x)J_{\nu+2n+k}(x)\,dx \\ &= \begin{cases} (2\nu+4n+2k)^{-1} & (m=n) \\ 0 \quad (m\neq n) \end{cases} \end{aligned}\right\}. \tag{2}$$

In the results (1) and (2) above, ν and k have been taken to be real, but this is not essential. However, for the applications in view, there is no need to consider complex values of these numbers, *and ν and k will be assumed to be real* throughout this note.

Two series analogous to the above two infinite integrals are likely to play an equally important role in what may be termed 'dual Fourier–Bessel series' and it is the purpose of this note to investigate them.

The first series, analogous to the integral I_1, is

$$S_1(\nu,m,k,r) \equiv \sum_{s=1}^{\infty} \frac{J_{\nu+2m+k}(\alpha_s)J_\nu(\alpha_s r)}{\alpha_s^k J_{\nu+1}^2(\alpha_s a)}, \tag{3}$$

where the α_s are the positive roots of $J_\nu(\alpha_s a) = 0$ and $a > 1$.

Applying Hankel's inversion formula to (1), we see that, if $\nu > -1$,

$$x^{-k}J_{\nu+2m+k}(x) = \frac{\Gamma(\nu+m+1)}{2^{k-1}\Gamma(\nu+1)\Gamma(m+k)}\int_0^1 r^{\nu+1}(1-r^2)^{k-1}\mathscr{F}_m(k+\nu,\nu+1,r^2)J_\nu(xr)\,dr. \tag{4}$$

The Fourier–Bessel expansion of the function $f(r)$ defined by

$$f(r) = \begin{cases} \dfrac{\Gamma(\nu+m+1)}{2^{k-1}\Gamma(\nu+1)\Gamma(m+k)} r^\nu(1-r^2)^{k-1}\mathscr{F}_m(k+\nu,\nu+1,r^2) & (0 < r < 1) \\ 0 \quad (1 < r < a) \end{cases} \tag{5}$$

is

$$f(r) = \sum_{s=1}^{\infty} A_s J_\nu(\alpha_s r), \tag{6}$$

where

$$A_s = \frac{2}{a^2 J_{\nu+1}^2(\alpha_s a)}\int_0^a rf(r)J_\nu(\alpha_s r)\,dr. \tag{7}$$

Substitution of $f(r)$ from (5) in (7) and use of (4) yields

$$A_s = \frac{2J_{\nu+2m+k}(\alpha_s)}{a^2J_{\nu+1}^2(\alpha_s a)\alpha_s^k}.$$

The first of the required results then follows from this value of A_s and equations (1), (3), (5), and (6). It is:

if m is zero or a positive integer, $\nu > -1$, $k > 0$, and α_s $(s = 1,2,3,...)$ are the positive roots of $J_\nu(\alpha_s a) = 0$ $(a > 1)$, then

$$\sum_{s=1}^{\infty} \frac{J_{\nu+2m+k}(\alpha_s)J_\nu(\alpha_s r)}{\alpha_s^k J_{\nu+1}^2(\alpha_s a)} = \tfrac{1}{2}a^2\int_0^\infty x^{1-k}J_{\nu+2m+k}(x)J_\nu(rx)\,dx, \tag{8}$$

both when $0 < r < 1$ and when $1 < r < a$.

The second series to be investigated is that analogous to the integral I_2 and is

$$S_2(\nu,m,n,k) \equiv \sum_{s=1}^{\infty} \frac{J_{\nu+2m+k}(\alpha_s)J_{\nu+2n+k}(\alpha_s)}{\alpha_s^2 J_{\nu+1}^2(\alpha_s a)}. \tag{9}$$

This series is evaluated by writing

$$F(z) = \frac{J_\nu(az)+iY_\nu(az)}{zJ_\nu(az)}\Phi(z) \tag{10}$$

where

$$\Phi(z) = J_{\nu+2m+k}(z)J_{\nu+2n+k}(z), \tag{11}$$

and considering $\int F(z)\,dz$ round a contour C. The contour C consists of

(i) the portions of the positive real axis joining the points

$$\delta,\ \alpha_1-\delta_1;\ \alpha_s+\delta_s,\ \alpha_{s+1}-\delta_{s+1}\ (s = 1,2,3,..., p-1);\ \alpha_p+\delta_p,\ \rho,$$

where the δ's are small and p, ρ are large and such that $\alpha_p < \rho < \alpha_{p+1}$;

(ii) a series of small semicircles γ_s $(s = 1, 2, 3,..., p)$ above the real axis with centres $z = \alpha_s$ and radii δ_s;

(iii) a large circular quadrant, with centre at the origin and radius ρ, extending from $z = \rho$ to $z = \rho e^{\frac{1}{2}i\pi}$;

(iv) the positive imaginary axis from $z = \rho e^{\frac{1}{2}i\pi}$ to $z = \delta e^{\frac{1}{2}i\pi}$;

(v) a small circular quadrant, with centre at the origin and radius δ, extending from $z = \delta e^{\frac{1}{2}i\pi}$ to $z = \delta$.

This contour avoids all the singularities of $F(z)$ and, if $I_1, I_2,..., I_5$ denote $\int F(z)\,dz$ taken along the corresponding portions of the contour C, then

$$I_1+I_2+I_3+I_4+I_5 = 0. \tag{12}$$

Now
$$I_1 = \int\limits_{\delta}^{\alpha_1-\delta_1} F(x)\,dx+\sum_{s=1}^{p-1}\int\limits_{\alpha_s+\delta_s}^{\alpha_{s+1}-\delta_{s+1}} F(x)\,dx+\int\limits_{\alpha_p+\delta_p}^{\rho} F(x)\,dx, \tag{13}$$

and
$$I_2 = \sum_{s=1}^{p}\int\limits_{\gamma_s} F(z)\,dz. \tag{14}$$

With $a > 1$, it can be shown that the integral I_3 along the large quadrant tends to zero as the radius ρ of the quadrant tends to infinity, and

$$I_4 = e^{\frac{1}{2}i\pi}\int\limits_{\rho}^{\delta} F(ye^{\frac{1}{2}i\pi})\,dy. \tag{15}$$

Finally, with the conditions specified for the integral in (2) and with the additional restrictions that $k > -m-n$ and that ν is not a negative integer, it can be shown that the integral I_5 along the small quadrant tends to zero as the radius δ of the quadrant tends to zero. Hence, substituting from (13), (14), and (15) in (12) and taking the limit as δ tends to zero and p, ρ tend to infinity, we get

$$\int\limits_{0}^{\alpha_1-\delta_1} F(x)\,dx+\sum_{s=1}^{\infty}\int\limits_{\alpha_s+\delta_s}^{\alpha_{s+1}-\delta_{s+1}} F(x)\,dx+$$
$$+\sum_{s=1}^{\infty}\int\limits_{\gamma_s} F(z)\,dz+e^{\frac{1}{2}i\pi}\int\limits_{\infty}^{0} F(ye^{\frac{1}{2}i\pi})\,dy = 0. \tag{16}$$

Since
$$J_\nu(aye^{\frac{1}{2}i\pi})+iY_\nu(aye^{\frac{1}{2}i\pi}) = \frac{2}{\pi i}e^{-\frac{1}{2}i\nu\pi}K_\nu(ay)$$

and
$$J_\nu(aye^{\frac{1}{2}i\pi}) = e^{\frac{1}{2}i\nu\pi}I_\nu(ay),$$

and, since the real part of $F(x)$ is $\Phi(x)/x$, it follows by equating real parts in (16), using (10), (11), and making a little reduction that

$$\mathrm{re}\left\{\sum_{s=1}^{\infty}\int_{\gamma_s} F(z)\,dz\right\} = -\int_0^{\alpha_1-\delta_1} x^{-1}\Phi(x)\,dx - \sum_{s=1}^{\infty}\int_{\alpha_s+\delta_s}^{\alpha_{s+1}-\delta_{s+1}} x^{-1}\Phi(x)\,dx +$$

$$+(-1)^{m+n}\frac{2}{\pi}\sin k\pi\int_0^{\infty}\frac{K_\nu(ay)}{yI_\nu(ay)}I_{\nu+2m+k}(y)I_{\nu+2n+k}(y)\,dy. \qquad (17)$$

If R_s is the residue of $F(z)$ at $z = \alpha_s$, $F(\alpha_s+\delta_s e^{i\theta})\delta_s e^{i\theta}$ tends to R_s uniformly with respect to θ as δ_s tends to zero, and hence

$$\lim_{\delta_s\to 0}\int_{\gamma_s} F(z)\,dz = \int_{\pi}^{0} R_s\,id\theta = -\pi i R_s. \qquad (18)$$

Now, since $\qquad J_\nu(z)Y_{\nu+1}(z)-J_{\nu+1}(z)Y_\nu(z) = -2/(\pi z)$

and $J_\nu(\alpha_s a) = 0$, it follows that

$$Y_\nu(\alpha_s a) = 2/\{\pi\alpha_s aJ_{\nu+1}(\alpha_s a)\},$$

and, remembering that $J'_\nu(\alpha_s a) = -J_{\nu+1}(\alpha_s a)$, we see that the residue R_s at $z = \alpha_s$ is given by

$$R_s = \frac{iY_\nu(\alpha_s a)}{\alpha_s aJ'_\nu(\alpha_s a)}\Phi(\alpha_s) = -\frac{2i\Phi(\alpha_s)}{\pi a^2\alpha_s^2 J_{\nu+1}^2(\alpha_s a)}. \qquad (19)$$

Substituting from (18) and (19) in (17), letting δ_s $(s = 1, 2, 3,\ldots)$ tend to zero, multiplying by $\frac{1}{2}a^2$ and substituting for Φ we obtain the second of the required results:

if m, n are zero or positive integers, $\nu > -m-n-k$ and is not a negative integer, $k > -m-n$ and α_s $(s = 1, 2, 3,\ldots)$ are the positive roots of $J_\nu(\alpha_s a) = 0$ $(a > 1)$, then

$$\sum_{s=1}^{\infty}\frac{J_{\nu+2m+k}(\alpha_s)J_{\nu+2n+k}(\alpha_s)}{\alpha_s^2 J_{\nu+1}^2(\alpha_s a)} = \tfrac{1}{2}a^2\int_0^{\infty} x^{-1}J_{\nu+2m+k}(x)J_{\nu+2n+k}(x)\,dx -$$

$$-(-1)^{m+n}\frac{a^2}{\pi}\sin k\pi\int_0^{\infty}\frac{K_\nu(ay)}{yI_\nu(ay)}I_{\nu+2m+k}(y)I_{\nu+2n+k}(y)\,dy. \qquad (20)$$

It will be observed from (8) that the sum of the series S_1 is always simply related to the value of the integral I_1, and equation (20) shows that a similar relation between S_2 and I_2 exists when k is an integer.

When k is not an integer, equations (2) and (20) provide a satisfactory means of computing the sum of the series S_2 especially when $a \gg 1$.

I am indebted to Dr. W. N. Everitt for some helpful conversations.

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SOME REMARKS ON STRONG SUMMABILITY

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1. This paper contains some remarks on the strong summability of series and some related topics. In §§ 2–13 we consider the general theory of strong summability, with particular reference to Abelian and Tauberian theorems. The most novel feature of this part of the paper is the introduction of strong Abel summability, which complements strong Cesàro summability in the same manner that ordinary Abel summability complements ordinary Cesàro summability. In §§ 14–15 we consider the relation of strong summability to absolute summability with index k and also the relation of the latter to the finiteness of a certain integral studied by Marcinkiewicz and Zygmund.

It is intended that the paper should give an account of strong summability which is reasonably complete in itself. At the same time it is hoped that the paper will serve as a partial introduction to a forthcoming paper (**8**) on the summability of a power series on its circle of convergence.

2. Notation. Let α be any real number other than a negative integer, and let

$$E_n^\alpha = \binom{\alpha+n}{n} = \frac{(\alpha+1)(\ldots)(\alpha+n)}{n!} \quad (n > 0), \qquad E_0^\alpha = 1.$$

For any given series $\sum a_n$ and any $n \geqslant 0$ we write

$$\sigma_n^0 = A_n^0 = a_0+a_1+\ldots+a_n, \qquad A_n^\alpha = \sum_{\nu=0}^{n} E_{n-\nu}^{\alpha-1} A_\nu^0,$$

$$\tau_n^0 = T_n^0 = na_n, \qquad T_n^\alpha = \sum_{\nu=1}^{n} E_{n-\nu}^{\alpha-1} T_\nu^0,$$

$$\sigma_n^\alpha = A_n^\alpha/E_n^\alpha, \qquad \tau_n^\alpha = T_n^\alpha/E_n^\alpha.$$

Then for any α and δ we have

$$A_n^{\alpha+\delta} = \sum_{\nu=0}^{n} E_{n-\nu}^{\delta-1} A_\nu^\alpha, \qquad T_n^{\alpha+\delta} = \sum_{\nu=1}^{n} E_{n-\nu}^{\delta-1} T_\nu^\alpha.$$

Also [Kogbetliantz (**14**)]

$$\tau_n^\alpha = n(\sigma_n^\alpha - \sigma_{n-1}^\alpha), \tag{2.1}$$

$$\tau_n^\alpha = -\alpha(\sigma_n^\alpha - \sigma_n^{\alpha-1}). \tag{2.2}$$

Quart. J. Math. Oxford (10) (1959), 115-39.

Further, if $\phi(x) = \sum_0^\infty a_n x^n$, the series being supposed convergent for $|x| < 1$, then

$$\phi(x) = (1-x)^{\alpha+1} \sum_0^\infty E_n^\alpha \sigma_n^\alpha x^n \quad (|x| < 1), \tag{2.3}$$

$$x\phi'(x) = (1-x)^{\alpha} \sum_1^\infty E_n^\alpha \tau_n^\alpha x^n \quad (|x| < 1). \tag{2.4}$$

For any number k used as an index (exponent) and such that $k > 1$, we write $k' = k/(k-1)$, so that k and k' are conjugate indices in the sense of Hölder's inequality.

We use $A(b, c, \ldots)$ to denote a positive constant depending only on b, c,..., not necessarily the same on any two occurrences. Inequalities of the form

$$L \leqslant A(b, c, \ldots) R$$

are to be interpreted as meaning 'if the expression R is finite, then the expression L is finite and satisfies the inequality'.

3. We say that a series $\sum a_n$ is 'strongly summable $(C, \alpha+1)$ with index k, or summable $\{\mathbf{C}, \alpha\}_k$, to the sum s' if

$$\sum_0^m |\sigma_n^\alpha - s|^k = o(m) \tag{3.1}$$

as $m \to \infty$.† We say also that $\sum a_n$ is bounded $\{\mathbf{C}, \alpha\}_k$ if

$$\sum_0^m |\sigma_n^\alpha|^k = O(m)$$

as $m \to \infty$. Summability and boundedness $\{\mathbf{C}, \alpha\}_k$ have been discussed by Winn (**21**) for $k = 1$, and by Hyslop (**13**), Chow (**2**), and the author (**5**) for general k. They are of little interest if $k < 1$‡ or $\alpha \leqslant -1$, and accordingly we shall suppose throughout that $k \geqslant 1$ and $\alpha > -1$. If $k \geqslant 1$ and (3.1) holds for some $\alpha > -1$, we say that $\sum a_n$ is 'summable $\{\mathbf{C}\}_k$', and similarly in the case of boundedness.

We may also define strong Abel summability, which corresponds to summability $\{\mathbf{C}, \alpha\}_k$ as summability (A) corresponds to summability (C, α). Thus we shall say that $\sum a_n$ is 'summable $\{\mathbf{A}\}_k$ to the sum s' if the series $\sum_0^\infty a_n x^n$ converges for $|x| < 1$ and its sum-function $\phi(x)$

† Summability $\{\mathbf{C}, \alpha\}_k$ is more usually called 'summability $[C, \alpha+1]_k$' or 'summability $[C; \alpha+1, k]$'. The notation used here, however, seems the more convenient.

‡ For instance, Kuttner (**16**) has shown that, if T is any regular Toeplitz method of summability, then, for any k such that $0 < k < 1$, there is a series which is not summable (T) but is summable $\{\mathbf{C}, 0\}_k$.

satisfies the condition

$$\int_0^R \frac{|\phi(x)-s|^k}{(1-x)^2}\,dx = o\left(\frac{1}{1-R}\right)$$

as $R \to 1-$. We say also that '$\sum a_n$ is bounded $\{\mathbf{A}\}_k$' if

$$\int_0^R \frac{|\phi(x)|^k}{(1-x)^2}\,dx = O\left(\frac{1}{1-R}\right)$$

as $R \to 1-$.

It is evident from Minkowski's inequality that a series cannot be summable $\{\mathbf{C},\alpha\}_k$ or $\{\mathbf{A}\}_k$ to two different sums.

We note in passing that summability $\{\mathbf{A}\}_k$ to the sum s is equivalent to the condition that

$$|\phi(x)-s|^k \to 0 \quad (C,1)$$

as $x \to 1-$; this follows easily by integration by parts. Moreover, summability $\{\mathbf{C},\alpha\}_k$ is evidently equivalent to

$$|\sigma_n^\alpha - s|^k \to 0 \quad (C,1)$$

as $n \to \infty$.

The condition (3.1) is easily seen† to be equivalent to the condition that

$$\left\{m^{\gamma-1}\sum_m^\infty \frac{|\sigma_n^\alpha - s|^k}{(n+1)^\gamma}\right\}^{1/k} = o(1), \tag{3.2}$$

where γ is any fixed number such that $\gamma > 1$. As $k \to \infty$ the expression on the left of (3.2) tends [cf. **(17)** 16] to

$$\sup_{n \geqslant m} |\sigma_n^\alpha - s|,$$

so that the limiting form of (3.2) as $k \to \infty$ is that

$$\sigma_n^\alpha - s = o(1).$$

Thus summability (C,α) may be regarded as the case $k = \infty$ of summability $\{\mathbf{C},\alpha\}_k$.

It is necessary to transform (3.1) into (3.2) in order to obtain a reasonable definition of summability $\{\mathbf{C},\alpha\}_k$ for $k = \infty$. If we take the $(1/k)$th power of both sides of (3.1) and make $k \to \infty$, we obtain formally

$$\sup_{n \leqslant m} |\sigma_n^\alpha - s| = o(1),$$

and this implies that $\sigma_n^\alpha = s$ for all n.

Similarly, summability (A) may be regarded as the case $k = \infty$ of summability $\{\mathbf{A}\}_k$. We may also regard boundedness (C,α) and boundedness (A) as the cases $k = \infty$ of boundedness $\{\mathbf{C},\alpha\}_k$ and boundedness $\{\mathbf{A}\}_k$, respectively.‡

† By partial summation.

‡ No preliminary transformation such as (3.2) is necessary here.

4. We may also consider sums involving the expression τ_n^α. Thus we shall say that $\sum a_n$ is 'summable $\{C,\alpha\}_k$', where $k \geqslant 1$ and $\alpha > -1$, if

$$\sum_1^m |\tau_n^\alpha|^k = o\,(m) \tag{4.1}$$

as $m \to \infty$. We may also regard the condition

$$\tau_n^\alpha = o\,(1) \tag{4.2}$$

as the case $k = \infty$ of summability $\{C,\alpha\}_k$. We say also that $\sum a_n$ is 'bounded $\{C,\alpha\}_k$' if (4.1), or (4.2), holds with the o replaced by O.

Since $-\tau_n^\alpha$ is the $(n-1)$th (C,α) mean of the series $\sum\limits_0^\infty \Delta(na_n)$, the summability $\{C,\alpha\}_k$ of $\sum a_n$ is simply the summability $\{\mathbf{C},\alpha\}_k$ of $\sum \Delta(na_n)$ to the sum 0, and similarly in the case of boundedness. It is more convenient, however, to refer to the property (4.1) and the corresponding O-relation as properties of the original series $\sum a_n$ than as properties of the series $\sum \Delta(na_n)$.

If (4.1), or (4.2), holds for some k and α, we say that $\sum a_n$ is 'summable $\{C\}_k$', and similarly in the case of boundedness.

The Abel-type property corresponding to summability $\{C\}_k$, which we shall call 'summability $\{A\}_k$', is that

$$\int_0^R (1-x)^{k-2}|\phi'(x)|^k\,dx = o\left(\frac{1}{1-R}\right) \tag{4.3}$$

as $R \to 1-$, the case $k = \infty$ here being that

$$(1-x)\phi'(x) = o\,(1) \tag{4.4}$$

as $x \to 1-$. We say also that $\sum a_n$ is 'bounded $\{A\}_k$' if (4.3), or (4.4), holds with the o replaced by O. It is easy to see that the summability $\{A\}_k$ of $\sum a_n$ is equivalent to the summability $\{\mathbf{A}\}_k$ of $\sum \Delta(na_n)$ to the sum 0.

I mention here that a different definition of 'strong Abel summability' has been given by Harington and Hyslop (**12**).† Their definition is equivalent to summability $\{A\}_k$ as defined above together with summability (A).

I mention also that Boyd and Hyslop (**1**) have given a definition of strong Rieszian summability in the particular case $\lambda_n = n$ and have proved that this is equivalent to strong Cesàro summability. A different definition of strong Rieszian summability, for general λ_n, has been given

† I am indebted to the referee for this and other references.

by Glatfeld (9). It follows from Glatfeld's Theorem 7 and the result of Boyd and Hyslop mentioned above that Glatfeld's summability $|R,n,\alpha|^k$ is equivalent to summability $\{\mathbf{C},\alpha\}_k$ for $\alpha > -1/k$.

5. In the following theorems I enumerate the various implications between summabilities $\{\mathbf{C},\alpha\}_k$, $\{\mathbf{C},\beta\}_r$, and $\{\mathbf{A}\}_l$, and between summabilities $\{C,\alpha\}_k$, $\{C,\beta\}_r$, and $\{A\}_l$. We have first a number of elementary results which can be collected together as

THEOREM 1. (i) *If $\sum a_n$ is summable $\{\mathbf{C},\alpha\}_k$ to the sum s, where $1 \leqslant k \leqslant \infty$ and $\alpha > -1$, it is summable $\{\mathbf{C},\alpha\}_r$ to s for every $r < k$.*

(ii) *If $\sum a_n$ is summable $\{\mathbf{A}\}_k$ to the sum s, where $1 \leqslant k \leqslant \infty$, it is summable $\{\mathbf{A}\}_r$ to s for every $r < k$.*

(iii) *If $0 < r < k < \infty$ and $\alpha > -1$, then for any s*†

$$\left\{\frac{1}{m+1}\sum_0^m |\sigma_n^\alpha - s|^r\right\}^{1/r} \leqslant \left\{\frac{1}{m+1}\sum_0^m |\sigma_n^\alpha - s|^k\right\}^{1/k} \leqslant \sup_{n\leqslant m}|\sigma_n^\alpha - s| \tag{5.1}$$

and

$$\left\{(1-R)\int_0^R \frac{|\phi(x)-s|^r}{(1-x)^2}\,dx\right\}^{1/r} \leqslant \left\{(1-R)\int_0^R \frac{|\phi(x)-s|^k}{(1-x)^2}\,dx\right\}^{1/k}$$
$$\leqslant \sup_x|\phi(x)-s| \leqslant \sup_n|\sigma_n^\alpha - s|. \tag{5.2}$$

(iv) *Throughout* (i)–(iii) *we may replace* $\mathbf{C}$ *by* C, $\mathbf{A}$ *by* A *(with omission of the sum s), $\sigma_n^\alpha - s$ by τ_n^α and $\phi(x)-s$ by $(1-x)\phi'(x)$.*

The first inequalities in (5.1) and (5.2) are simple consequences of Hölder's inequality. The remaining results are either immediate or follow from the identity (2.3).

In the direction of increasing k we have more complicated results.

THEOREM 2. *Suppose that $\alpha > -1$ and that either $1 < k \leqslant r < \infty$ and $\beta \geqslant \alpha + 1/k - 1/r$ or $1 = k \leqslant r < \infty$ and $\beta > \alpha + 1/k - 1/r$. Then*

(i) *if $\sum a_n$ is summable $\{\mathbf{C},\alpha\}_k$ to the sum s, it is summable $\{\mathbf{C},\beta\}_r$ to s,*

(ii) *for any s*

$$\sup_m\left\{\frac{1}{m+1}\sum_0^m |\sigma_n^\beta - s|^r\right\}^{1/r} \leqslant A(k,r,\alpha,\beta)\sup_m\left\{\frac{1}{m+1}\sum_0^m |\sigma_n^\alpha - s|^k\right\}^{1/k},$$

(iii) *in* (i) *and* (ii) *we may replace* $\mathbf{C}$ *by* C, *$\sigma_n^\alpha - s$ by τ_n^α, and $\sigma_n^\beta - s$ by τ_n^β.*

† These inequalities are, of course, the analogues of (i) and (ii) for boundedness $\{\mathbf{C}\}_k$ and $\{\mathbf{A}\}_k$.

THEOREM 3. *Suppose that* $\alpha > -1$ *and that either* $k > 1$ *and* $\beta > \alpha + 1/k$ *or* $k = 1$ *and* $\beta \geqslant \alpha + 1$. *Then*

(i) *if* $\sum a_n$ *is summable* $\{\mathbf{C}, \alpha\}_k$ *to the sum* s, *it is summable* (C, β) *to* s,

(ii) *for any* s

$$\sup_m |\sigma_m^\beta - s| \leqslant A(k, \alpha, \beta) \sup_m \left\{\frac{1}{m+1} \sum_0^m |\sigma_n^\alpha - s|^k\right\}^{1/k},$$

(iii) *if* $\sum a_n$ *is summable* $\{C, \alpha\}_k$, $\tau_n^\beta = o(1)$,

(iv) $$\sup_m |\tau_m^\beta| \leqslant A(k, \alpha, \beta) \sup_m \left\{\frac{1}{m+1} \sum_1^m |\tau_n^\alpha|^k\right\}^{1/k}.$$

The case $r = k = 1$ of Theorem 2 is due to Winn (**21**), while the case $r = k > 1$ is due to Hyslop (**13**) [see also Chow (**2**)]. The theorem for general k and r was proved by the author in (**5**) and (**6**) under the extra conditions $\alpha > 0$ and $\alpha > -1/k'$, respectively.† The general result may be deduced from these special cases by a device due to Chow [see (**2**), Lemmas 2 and 3]. Alternatively, it can be proved directly, and I give such a proof in § 7.

The case $k = 1$ of Theorem 3 is trivial. The cases $k > 1$, $\alpha = 0$ and $k > 1$, $\alpha > -1/k'$ are due to Kuttner (**16**) and Hyslop (**13**), respectively, while the general result is due to Chow (**2**). Since the proof is short and Chow's paper is not readily available, I give the proof of this also (see § 8).

There is also a theorem connecting summability $\{\mathbf{C}, \alpha\}_k$ with summability $\{\mathbf{A}\}_t$, and summability $\{C, \alpha\}_k$ with summability $\{A\}_t$.

THEOREM 4. *Suppose that* $1 \leqslant k \leqslant \infty$ *and* $\alpha > -1$. *Then*

(i) *if* $\sum a_n$ *is summable* $\{\mathbf{C}, \alpha\}_k$ *to the sum* s, *it is summable* $\{\mathbf{A}\}_t$ *to* s *for every* t,

(ii) *if* $\sum a_n$ *is summable* $\{C, \alpha\}_k$, *it is summable* $\{A\}_t$ *for every* t.

This, however, is immediate. By Theorem 3, summability $\{\mathbf{C}\}_k$ implies summability (C) and so summability (A), and this in turn implies summability $\{\mathbf{A}\}_t$ for every t (Theorem 1). This proves (i), and (ii) follows from (i) applied to $\sum \Delta(na_n)$.

We have also the following convexity theorem.

THEOREM 5. *Suppose that* $1 \leqslant k \leqslant \infty$ *and* $\alpha > -1$. *Then*

(i) *if* $\sum a_n$ *is bounded* $\{\mathbf{C}, \alpha\}_k$ *and summable* $\{\mathbf{C}\}_k$ (*or summable* (C)‡) *to the sum* s, *it is summable* $\{\mathbf{C}, \beta\}_k$ *to* s *for every* $\beta > \alpha$,

† The case $\alpha > -1/k$ is also contained in results of Glatfeld (**9**).

‡ By Theorems 1 and 3, $\sum a_n$ is summable $\{\mathbf{C}\}_k$ if and only if it is summable (C).

(ii) *if* $\sum a_n$ *is bounded* $\{C,\alpha\}_k$ *and summable* $\{C\}_k$, *it is summable* $\{C,\beta\}_k$ *for every* $\beta > \alpha$.

The case $k = \infty$ of (i) is a well-known theorem in the theory of ordinary Cesàro summability [see Kogbetliantz (**15**) 24]. The case $k = 1$ is due to Winn (**21**), while the general case is stated by Chow (**3**)† with the remark that it can be proved by an argument similar to Winn's. I give in § 9 an alternative argument which, although similar in principle to that of Winn, is considerably simpler in detail.

There are no implications between summabilities $\{\mathbf{C},\alpha\}_k$, $\{\mathbf{C},\beta\}_r$, and $\{\mathbf{A}\}_l$, and between summabilities $\{C,\alpha\}_k$, $\{C,\beta\}_r$, and $\{A\}_l$, other than those contained, explicitly or implicitly, in Theorems 1–4. The proofs of the various negative results required to establish this are given in § 12.

6. We pass next to implications between different types of strong summability. We have first the following result [Hyslop (**13**), Chow (**2**)].

THEOREM 6. *Suppose that* $1 \leqslant k \leqslant \infty$ *and* $\alpha > -1$. *Then, if* $\sum a_n$ *is summable* $\{\mathbf{C},\alpha\}_k$ *to some s, it is summable* $\{C,\alpha+1\}_k$.

The corresponding Abel-type theorem is false, and in fact summability $\{\mathbf{A}\}_k$ does not imply summability $\{A\}_r$ for any k and r ($+\infty$ included). Moreover, the converse of Theorem 6 is false: summability $\{C,\alpha\}_k$ or $\{A\}_k$ does not imply summability $\{\mathbf{C},\beta\}_r$ or $\{\mathbf{A}\}_r$ for any k, r, α, and β. These negative results are proved in § 13.

Summability and boundedness $\{C\}_k$ are Tauberian conditions for summability $\{\mathbf{C}\}_k$, and this fact gives rise to a number of results. Thus we have

THEOREM 7. *Suppose that* $1 \leqslant k \leqslant \infty$ *and* $\alpha > -1$. *Then, if* $\sum a_n$ *is summable* $\{\mathbf{C}\}_k$ *to the sum s, or summable* (C) *to s, and is summable* $\{C,\alpha+1\}_k$, *then it is summable* $\{\mathbf{C},\alpha\}_k$ *to s.*

Theorem 7 is only of 'o' depth, the corresponding 'O' result being

THEOREM 8. *Suppose that* $1 \leqslant k \leqslant \infty$ *and* $\alpha > -1$. *Then, if* $\sum a_n$ *is summable* $\{\mathbf{C}\}_k$ *to the sum s, or summable* (C) *to s, and is bounded* $\{C,\alpha+1\}_k$, *it is summable* $\{\mathbf{C},\beta\}_k$ *to s for every* $\beta > \alpha$.

The cases $k = \infty$ of Theorems 6–8 are, of course, well-known results in the theory of ordinary Cesàro summability [see Kogbetliantz (**15**) 15, 30, 31].

We note as a particular consequence of Theorem 7 that, if $\sum a_n$ is

† I am indebted to Dr. L. S. Bosanquet for this information.

summable $\{\mathbf{C}\}_k$ and summable $\{C,\alpha\}_k$, then it is summable (C,α). Further, by Theorem 8, if $\sum a_n$ is summable $\{\mathbf{C}\}_k$ and bounded $\{C,\alpha\}_k$, where $k>1$, then it is summable (C,α), while, if $k=1$, it is summable $\{\mathbf{C},\alpha\}_r$ for every finite r. The following theorem gives the Abel analogue of these results.

THEOREM 9. (i) *If $\sum a_n$ is summable $\{\mathbf{A}\}_k$ to s, and is also summable $\{A\}_k$, where $k \geqslant 1$, then it is summable (A) to s.*

(ii) *If $\sum a_n$ is summable $\{\mathbf{A}\}_k$ to s, and is also bounded $\{A\}_k$, where $k>1$, then it is summable (A) to s.*

(iii) *If $\sum a_n$ is summable $\{\mathbf{A}\}_1$ to s, and is also bounded $\{A\}_1$, then it is summable $\{\mathbf{A}\}_r$ to s for every finite r.*

We have also generalizations of the 'o' and 'O' Tauberian theorems for summability (A).

THEOREM 10. *Suppose that $1 \leqslant k \leqslant \infty$ and $\alpha > -1$. Then, if $\sum a_n$ is summable $\{\mathbf{A}\}_k$ to the sum s, and is also summable $\{C,\alpha+1\}_k$, it is summable $\{\mathbf{C},\alpha\}_k$ to s.*

THEOREM 11. *Suppose that $1 \leqslant k \leqslant \infty$ and $\alpha > -1$. Then, if $\sum a_n$ is summable $\{\mathbf{A}\}_k$ to the sum s, and is either bounded $\{\mathbf{C},\alpha\}_k$ or $\{C,\alpha+1\}_k$, it is summable $\{\mathbf{C},\beta\}_k$ to s for every $\beta>\alpha$.*

Theorems 10 and 11 are, of course, stronger than Theorems 7 and 8, respectively, and, as before, the cases $k=\infty$ are well known [see Kogbetliantz (**15**) 38–40]. Further, Theorem 10 can be deduced from Theorem 11, but the former is much more elementary than Theorem 11. The case $k<\infty$ of Theorem 10 has been proved by the author in (**7**), and I give here an alternative proof. Both theorems generalize a Tauberian theorem of the author (**5**).

We note also the inequality form of Theorems 10 and 11.

THEOREM 12. *Suppose that $1 \leqslant k < \infty$ and $\alpha > -1$. Then*

$$\sup_m \left\{\frac{1}{m+1}\sum_0^m |\sigma_n^\alpha|^k\right\}^{1/k} \leqslant A(k,\alpha)\sup_m\left\{\frac{1}{m+1}\sum_1^m |\tau_n^{\alpha+1}|^k\right\}^{1/k}+$$
$$+A(k)\sup_R\left\{(1-R)\int_0^R \frac{|\phi(x)|^k}{(1-x)^2}\,dx\right\}^{1/k}.$$

Theorem 12 can be proved by an argument similar to that of Theorem 10. There is also an alternative, given in (**7**).

I give the proof of Theorem 9 in § 10, and the proofs of the remaining results of this paragraph† in § 11.

† Always for $k<\infty$.

7. We pass now to the proofs of the results stated in the preceding paragraphs, viz. Theorem 2, the case $k > 1$ of Theorem 3, the cases $k < \infty$ of Theorems 5–11, and the various negative results mentioned in § 5 and § 6.

We begin with the proof of Theorem 2, and for this we require the following lemmas.

LEMMA 1. *Let $\mu < 1$, let $s_n \geqslant 0$, $t_n = (n+1)^{-1}\sum_0^n s_\nu$ and $t = \sup t_n$. Then*

$$\sum_{\nu=0}^{n} (\nu+1)^{-\mu} s_\nu \leqslant A(\mu)(n+1)^{1-\mu} t.$$

If also $t_n = o(1)$ as $n \to \infty$, then

$$\sum_{\nu=0}^{n} (\nu+1)^{-\mu} s_\nu = o(n^{1-\mu}).$$

This follows immediately from a partial summation.

LEMMA 2. *Let $1 < k < r < \infty$, $\delta = 1/k - 1/r$, $c_n \geqslant 0$, and*

$$C_n = \sum_{\nu<n} (n-\nu)^{\delta-1} c_\nu.$$

Then

$$\{\sum C_n^r\}^{1/r} \leqslant A(k,r)\{\sum c_n^k\}^{1/k}.$$

This is a well-known inequality given by Hardy, Littlewood, and Pólya (11).

Consider now the proof of Theorem 2. We may evidently restrict ourselves to (i) and (ii), and in these we may evidently suppose that $s = 0$. We then have to prove that

$$\sup_m \left\{\frac{1}{m+1}\sum_0^m |\sigma_n^{\alpha+\delta}|^r\right\}^{1/r} \leqslant A(k,r,\alpha,\delta) \sup_m \left\{\frac{1}{m+1}\sum_0^m |\sigma_n^{\alpha}|^k\right\}^{1/k}, \quad (7.1)$$

and that $\quad \sum_0^m |\sigma_n^\alpha|^k = o(m) \quad$ implies $\quad \sum_0^m |\sigma_n^{\alpha+\delta}|^r = o(m), \qquad (7.2)$

where $\alpha > -1$ and either (*a*) $1 \leqslant k \leqslant r < \infty$ and $\delta > 1/k - 1/r$, or (*b*) $1 < k < r < \infty$ and $\delta = 1/k - 1/r$.

Consider the case (*a*), and let us suppose first that $k > 1$. Let S denote the supremum on the right of (7.1), and let λ, μ, η be numbers depending on k, r, α, δ, to be chosen later, such that

$$\lambda < 1/k', \quad 0 < \eta < 1, \quad k'(\delta-1)(1-\eta) > -1, \quad 0 < \mu < 1.$$

Write also B for a constant depending on some or all of k, r, α, δ.

We observe now that

$$|\sigma_n^{\alpha+\delta}| = \left|\frac{1}{E_n^{\alpha+\delta}}\sum_{\nu=0}^{n} E_{n-\nu}^{\delta-1} E_\nu^\alpha \sigma_\nu^\alpha\right| \leqslant B(n+1)^{-\alpha-\delta}\sum_{\nu=0}^{n}(n+1-\nu)^{\delta-1}(\nu+1)^\alpha|\sigma_\nu^\alpha|$$

$$= B(n+1)^{-\alpha-\delta}\sum_{\nu=0}^{n}\{(n+1-\nu)^{(\delta-1)\eta}(\nu+1)^{\alpha+\lambda+\mu(r-k)/(kr)}|\sigma_\nu^\alpha|^{k/r}\}\times$$

$$\times\{(n+1-\nu)^{(\delta-1)(1-\eta)}(\nu+1)^{-\lambda}\}\{(\nu+1)^{-\mu}|\sigma_\nu^\alpha|^k\}^{(r-k)/(kr)}.$$

Applying Hölder's inequality with indices r, k', $kr/(r-k)$, we get

$$|\sigma_n^{\alpha+\delta}| \leqslant B(n+1)^{-\alpha-\delta}\left\{\sum_{\nu=0}^{n}(n+1-\nu)^{r(\delta-1)\eta}(\nu+1)^{r(\alpha+\lambda+\mu/k)-\mu}|\sigma_\nu^\alpha|^k\right\}^{1/r}\times$$

$$\times\left\{\sum_{\nu=0}^{n}(n+1-\nu)^{k'(\delta-1)(1-\eta)}(\nu+1)^{-k'\lambda}\right\}^{1/k'}\left\{\sum_{\nu=0}^{n}(\nu+1)^{-\mu}|\sigma_\nu^\alpha|^k\right\}^{(r-k)/(kr)}, \tag{7.3}$$

where the last factor on the right is to be omitted if $r = k$. Using Lemma 1 we obtain immediately

$$|\sigma_n^{\alpha+\delta}| \leqslant BS^{1-k/r}(n+1)^t\left\{\sum_{\nu=0}^{n}(n+1-\nu)^{r(\delta-1)\eta}(\nu+1)^{r(\alpha+\lambda+\mu/k)-\mu}|\sigma_\nu^\alpha|^k\right\}^{1/r},$$

where

$$t = -\alpha-\delta+(1-\mu)(r-k)/(kr)+(\delta-1)(1-\eta)-\lambda+1/k'.$$

Write now $\quad p = r(\delta-1)\eta, \qquad q = r(\alpha+\lambda+\mu/k),$

so that $rt = \mu-p-q-1$. Then, for $n \leqslant m$, we have

$$|\sigma_n^{\alpha+\delta}|^r \leqslant BS^{r-k}(n+1)^{\mu-p-q-1}\sum_{\nu=0}^{n}(n+1-\nu)^p(\nu+1)^{q-\mu}|\sigma_\nu^\alpha|^k$$

$$\leqslant BS^{r-k}(m+1)^\mu(n+1)^{-p-q-1}\sum_{\nu=0}^{n}(n+1-\nu)^p(\nu+1)^{q-\mu}|\sigma_\nu^\alpha|^k.$$

Hence, for $m \geqslant 0$,

$$\sum_{n=0}^{m}|\sigma_n^{\alpha+\delta}|^r \leqslant BS^{r-k}(m+1)^\mu\sum_{\nu=0}^{m}h_\nu(\nu+1)^{-\mu}|\sigma_\nu^\alpha|^k,$$

where

$$(\nu+1)^{-q}h_\nu = \sum_{n=\nu}^{m}(n+1-\nu)^p(n+1)^{-p-q-1} \leqslant \sum_{n=\nu}^{\infty} = \sum_{n=\nu}^{2\nu-1}+\sum_{n=2\nu}^{\infty}$$

$$\leqslant B(\nu+1)^{-p-q-1}\sum_{n=\nu}^{2\nu-1}(n+1-\nu)^p+B\sum_{n=\nu}^{\infty}(n+1)^{-q-1} \leqslant B(\nu+1)^{-q},$$

provided that $p > -1$ and $q > 0$. Hence

$$\sum_{n=0}^{m}|\sigma_n^{\alpha+\delta}|^r \leqslant BS^{r-k}(m+1)^\mu\sum_{\nu=0}^{m}(\nu+1)^{-\mu}|\sigma_\nu^\alpha|^k,$$

and (7.1) and (7.2) now follow from Lemma 1.

It remains to show that λ, μ, η can be chosen to satisfy the various conditions imposed. Choose η so that $r\eta = k'(1-\eta)$, i.e. so that

$\eta = k'/(r+k')$. Then $0 < \eta < 1$, and

$$\delta - 1 > \frac{1}{k} - \frac{1}{r} - 1 = -\left(\frac{1}{k'} + \frac{1}{r}\right) = -\frac{1}{r\eta} = -\frac{1}{k'(1-\eta)}.$$

Since also $\alpha > -1$, we may choose λ and μ such that

$$\lambda < 1/k', \quad 0 < \mu < 1, \quad \alpha+\lambda+\mu/k > 0.$$

Then $q > 0$, and all the required conditions are satisfied.

This proves case (a) when $k > 1$. A similar argument applies when $k = 1$: we take $\lambda = 0$, $\eta = 1$ and omit the last factor but one in (7.3).

We turn now to case (b). We have first

$$|\sigma_n^{\alpha+\delta}| \leqslant \frac{1}{E_n^{\alpha+\delta}} \sum_{0 \leqslant \nu \leqslant \frac{1}{2}n} E_{n-\nu}^{\delta-1} E_\nu^\alpha |\sigma_\nu^\alpha| + \frac{1}{E_n^{\alpha+\delta}} \sum_{\frac{1}{2}n \leqslant \nu < n} E_{n-\nu}^{\delta-1} E_\nu^\alpha |\sigma_\nu^\alpha| + \frac{E_n^\alpha |\sigma_n^\alpha|}{E_n^{\alpha+\delta}}$$
$$= P_n + Q_n + R_n, \quad \text{say.}$$

Hence, by Minkowski's inequality, it is enough to prove that, if W_n is any one of P_n, Q_n, R_n, then

$$\sup_m \left\{\frac{1}{m+1} \sum_{n=0}^{m} W_n^r\right\}^{1/r} \leqslant B \sup_m \left\{\frac{1}{m+1} \sum_{n=0}^{m} |\sigma_n^\alpha|^k\right\}^{1/k} \tag{7.4}$$

and
$$\sum_{n=0}^{m} |\sigma_n^\alpha|^k = o(m) \quad \text{implies} \quad \sum_{n=0}^{m} W_n^r = o(m). \tag{7.5}$$

Here
$$P_n \leqslant \frac{B}{E_n^{\alpha+1}} \sum_{0 \leqslant \nu \leqslant \frac{1}{2}n} E_\nu^\alpha |\sigma_\nu^\alpha| \leqslant \frac{B}{E_n^{\alpha+1}} \sum_{\nu=0}^{n} E_\nu^\alpha |\sigma_\nu^\alpha|,$$

and applying the arguments of case (a) (with $\delta = 1$) to the expression on the right we see easily that (7.4) and (7.5) are satisfied when $W_n = P_n$. Also

$$\left\{\sum_0^m R_n^r\right\}^{1/r} \leqslant B\left\{\sum_0^m (n+1)^{-r\delta} |\sigma_n^\alpha|^r\right\}^{1/r} \leqslant B\left\{\sum_0^m (n+1)^{-k\delta} |\sigma_n^\alpha|^k\right\}^{1/k},$$

and, since $k\delta = 1 - k/r < 1$, it follows from Lemma 1 that R_n also satisfies (7.4) and (7.5).

Write now

$$c_n = (n+1)^{-\delta} |\sigma_n^\alpha| \quad (n \leqslant m), \quad c_n = 0 \quad (n > m),$$
$$C_n = \sum_{\nu < n} (n-\nu)^{\delta-1} c_\nu.$$

Then $Q_n \leqslant BC_n$, whence, by Lemma 2,

$$\left\{\sum_0^m Q_n^r\right\}^{1/r} \leqslant B\left\{\sum_0^m C_n^r\right\}^{1/r} \leqslant B\left\{\sum_0^m c_n^k\right\}^{1/k} = B\left\{\sum_0^m (n+1)^{-k\delta} |\sigma_n^\alpha|^k\right\}^{1/k}.$$

Applying Lemma 1 once again, we obtain (7.4) and (7.5) with $W_n = Q_n$, and this completes the proof.

8. Consider next the case $k > 1$ of Theorem 3. Here also we may restrict ourselves to the σ-results and here again we may suppose that $s = 0$. Since

$$|\sigma_n^{\alpha+\delta}|^k \leqslant \left\{\frac{1}{E_n^{\alpha+\delta}} \sum_0^n E_{n-\nu}^{\delta-1} E_\nu^\alpha |\sigma_\nu^\alpha|\right\}^k$$

$$\leqslant B(n+1)^{-k\alpha-k\delta}\left\{\sum_0^n E_\nu^\alpha |\sigma_\nu^\alpha|^k\right\}\left\{\sum_0^n (E_{n-\nu}^{\delta-1})^{k'} E_\nu^\alpha\right\}^{k-1}$$

$$\leqslant B(n+1)^{-\alpha-1} \sum_0^n E_\nu^\alpha |\sigma_\nu^\alpha|^k,$$

provided that $\delta > 1/k$, and since $\alpha > -1$, the required result follows immediately from Lemma 1.

9. We turn next to the case $k < \infty$ of Theorem 5. We prove first the following lemma. (The argument used is a modification of one used by Mr. A. E. Ingham in his lectures to prove the case $k = \infty$.)

LEMMA 3. *Suppose that $1 \leqslant k < \infty$ and $\alpha > -1$. Then, if $\sum a_n$ is bounded $\{C, \alpha\}_k$ and summable $\{C, \alpha+1\}_k$ to the sum 0, it is summable $\{C, \alpha+\delta\}_k$ to 0 for every $\delta > 0$.*

By Theorem 2, we may suppose that $0 < \delta < 1/k$. For any integer $p > 0$ we have (formally)

$$\sum_{\nu=0}^\infty A_\nu^\alpha x^\nu = \sum_{\nu=p+1}^\infty A_\nu^\alpha x^\nu + (1-x)\sum_{\nu=0}^{p-1} A_\nu^{\alpha+1} x^\nu + A_p^{\alpha+1} x^p.$$

Multiplying by $(1-x)^{-\delta}$, expanding both sides in powers of x and equating the coefficients of x^n, where $n > p > 0$, we get

$$A_n^{\alpha+\delta} = \sum_{\nu=p+1}^n E_{n-\nu}^{\delta-1} A_\nu^\alpha + \sum_{\nu=0}^{p-1} E_{n-\nu}^{\delta-2} A_\nu^{\alpha+1} + E_{n-p}^{\delta-1} A_p^{\alpha+1},$$

and so (with $\mu = n-\nu$ in the first sum)

$$\sigma_n^{\alpha+\delta} = \frac{1}{E_n^{\alpha+\delta}} \sum_{\mu=0}^{n-p-1} E_\mu^{\delta-1} E_{n-\mu}^\alpha \sigma_{n-\mu}^\alpha +$$

$$+ \frac{1}{E_n^{\alpha+\delta}} \sum_{\nu=0}^{p-1} E_{n-\nu}^{\delta-2} E_\nu^{\alpha+1} \sigma_\nu^{\alpha+1} + \frac{E_{n-p}^{\delta-1} E_p^{\alpha+1} \sigma_p^{\alpha+1}}{E_n^{\alpha+\delta}}$$

$$= U+V+W, \quad \text{say}.$$

Now let θ be a fixed number such that $0 < \theta < \frac{1}{2}$, let $q_n = [\theta n]$, and let $p = p_n = n-q_n$, so that U, V, W are functions of n, say U_n, V_n, W_n. If, in addition, $n \geqslant 1/\theta$, then clearly $n > p > 0$.

In U_n we have
$$0 \leqslant \mu \leqslant q_n - 1 < \theta n,$$
so that
$$|U_n| \leqslant \frac{1}{E_n^{\alpha+\delta}} \sum_{0 \leqslant \mu < \theta n} E_\mu^{\delta-1} E_{n-\mu}^{\alpha} |\sigma_{n-\mu}^{\alpha}| \leqslant A(\alpha, \delta) \sum_{0 \leqslant \mu < \theta n} E_\mu^{\delta-1} E_{n-\mu}^{-\delta} |\sigma_{n-\mu}^{\alpha}|.$$

Let now
$$S = \sup_m \left\{ \frac{1}{m+1} \sum_0^m |\sigma_n^\alpha|^k \right\}^{1/k}.$$

Then, by Minkowski's inequality and Lemma 1,
$$\begin{aligned}
\sum_{1/\theta \leqslant n \leqslant m} |U_n|^k &\leqslant A(k, \alpha, \delta) \sum_{n=0}^m \Big\{ \sum_{0 \leqslant \mu < \theta n} E_\mu^{\delta-1} E_{n-\mu}^{-\delta} |\sigma_{n-\mu}^\alpha| \Big\}^k \\
&\leqslant A(k, \alpha, \delta) \Big\{ \sum_{0 \leqslant \mu < \theta m} E_\mu^{\delta-1} \Big(\sum_{\mu/\theta < n \leqslant m} (E_{n-\mu}^{-\delta})^k |\sigma_{n-\mu}^\alpha|^k \Big)^{1/k} \Big\}^k \\
&\leqslant A(k, \alpha, \delta) \Big\{ \sum_{0 \leqslant \mu < \theta m} E_\mu^{\delta-1} \Big\}^k \Big\{ \sum_{n=0}^m (n+1)^{-k\delta} |\sigma_n^\alpha|^k \Big\} \\
&\leqslant A(k, \alpha, \delta) S^k \theta^{k\delta} (m+1)
\end{aligned}$$
(since $k\delta < 1$), the constant being independent of θ.

In V_n we have
$$\theta n < q_n + 1 \leqslant n - \nu \leqslant n.$$
If then $k > 1$, Hölder's inequality gives
$$\begin{aligned}
|V_n|^k &\leqslant (E_n^{\alpha+\delta})^{-k} \Big\{ \sum_{\nu=0}^{p-1} |E_{n-\nu}^{\delta-2}|^{k'} \Big\}^{k/k'} \Big\{ \sum_{\nu=0}^{p-1} (E_\nu^{\alpha+1})^k |\sigma_\nu^{\alpha+1}|^k \Big\} \\
&= O(n^{-k\alpha-k-1})\, o(n^{1+k\alpha+k}) = o(1),
\end{aligned}$$
and so
$$\sum_{1/\theta \leqslant n \leqslant m} |V_n|^k = o(m). \tag{9.1}$$

On the other hand, if $k = 1$,
$$V_n = O\Big(n^{-\alpha-2} \sum_{\nu=0}^{p-1} E_\nu^{\alpha+1} |\sigma_\nu^{\alpha+1}| \Big) = O(n^{-\alpha-2})\, o(n^{\alpha+2}) = o(1),$$
so that (9.1) holds for all $k \geqslant 1$.

We have also
$$|W_n| \leqslant A(\alpha, \delta, \theta) |\sigma_p^{\alpha+1}|.$$
Since $0 < \theta < \frac{1}{2}$, there are at most two successive values of n for which p_n remains constant. Hence
$$\sum_{1/\theta \leqslant n \leqslant m} |W_n|^k \leqslant A(k, \alpha, \delta, \theta) \sum_0^m |\sigma_n^{\alpha+1}|^k = o(m).$$
It follows now by Minkowski's inequality that
$$\limsup_{m \to \infty} \left\{ \frac{1}{m+1} \sum_{n=0}^m |\sigma_n^{\alpha+\delta}|^k \right\}^{1/k} \leqslant A(k, \alpha, \delta) S \theta^\delta. \tag{9.2}$$
Since θ can be as small as we please, it follows that the limit superior on the left of (9.2) is 0, and this is the required result.

Consider now the proof of Theorem 5 (for $k < \infty$, of course). We may restrict ourselves once again to the σ-results and may again suppose that $s = 0$. If $\sum a_n$ is summable $\{\mathbf{C}\}_k$, it is summable $\{\mathbf{C}, \alpha+\gamma\}_k$ for some integer γ [Theorem 2 (i)]. If also it is bounded $\{\mathbf{C}, \alpha\}_k$, then it is bounded $\{\mathbf{C}, \beta\}_k$ for every $\beta \geqslant \alpha$ [Theorem 2 (ii)], and the result of the theorem now follows by repeated use of Lemma 3.

10. We consider next Theorem 9. We observe first that $\phi(x)-s$ vanishes at only a finite number of points in $0 \leqslant x \leqslant R < 1$ (since ϕ is a power series), so that $|\phi(x)-s|$ is differentiable in $(0, R)$ except at a finite number of points. Hence, by integration by parts, we have

$$\int\limits_0^R \frac{|\phi(x)-s|^k}{(1-x)^2}\,dx = \left[\frac{|\phi(x)-s|^k}{1-x}\right]_0^R - k\int\limits_0^R \frac{|\phi(x)-s|^{k-1}}{1-x}\,\frac{d}{dx}|\phi(x)-s|\,dx \tag{10.1}$$

for any $k \geqslant 1$. Since

$$\left|\frac{d}{dx}|\phi(x)-s|\right| \leqslant \left|\frac{d}{dx}\{\phi(x)-s\}\right| = |\phi'(x)|$$

whenever the left-hand side exists, (10.1) gives

$$|\phi(R)-s|^k \leqslant (1-R)|\phi(0)-s|^k +$$

$$+(1-R)\int\limits_0^R \frac{|\phi(x)-s|^k}{(1-x)^2}\,dx + (1-R)\int\limits_0^R |\phi(x)-s|^{k-1}\frac{|\phi'(x)|}{(1-x)}\,dx. \tag{10.2}$$

It follows immediately from this last result that, if $\sum a_n$ is summable $\{\mathbf{A}\}_1$ to s and summable $\{A\}_1$, then it is summable (A) to s. It follows also that, if it is summable $\{\mathbf{A}\}_1$ to s and bounded $\{A\}_1$, then it is bounded (A), and this together with summability $\{\mathbf{A}\}_1$ implies summability $\{\mathbf{A}\}_r$ for every finite r, for, if $1 \leqslant r < \infty$,

$$\int\limits_0^R \frac{|\phi(x)-s|^r}{(1-x)^2}\,dx \leqslant \left\{\sup_{0\leqslant x\leqslant R} |\phi(x)-s|^{r-1}\right\}\int\limits_0^R \frac{|\phi(x)-s|}{(1-x)^2}\,dx = o\left(\frac{1}{1-R}\right).$$

It remains to be shown that, if $1 < k < \infty$, then summability $\{\mathbf{A}\}_k$ and boundedness $\{A\}_k$ together imply summability (A). By Hölder's inequality, we have

$$(1-R)\int\limits_0^R |\phi(x)-s|^{k-1}\frac{|\phi'(x)|}{(1-x)}\,dx$$

$$\leqslant \left\{(1-R)\int\limits_0^R \frac{|\phi(x)-s|^k}{(1-x)^2}\,dx\right\}^{1/k'}\left\{(1-R)\int\limits_0^R (1-x)^{k-2}|\phi'(x)|^k\,dx\right\}^{1/k},$$

and the expression on the right is $o(1)$ if $\sum a_n$ is summable $\{\mathbf{A}\}_k$ and bounded $\{A\}_k$. Combining this with (10.2) we obtain the desired result.

11. We can now prove the remaining results of § 6 (always for $k < \infty$). To prove Theorems 6 and 7 we note that, if $\sum a_n$ is summable $\{\mathbf{C}, \alpha+1\}_k$, then a necessary and sufficient condition that it should be summable $\{\mathbf{C}, \alpha\}_k$ is that it be summable $\{C, \alpha+1\}_k$: this is a simple consequence of the identity (2.2) and Minkowski's inequality. Theorem 6 follows immediately from this remark and the case $k = r$ of Theorem 2 (i), while Theorem 7 follows by repeated use of this together with the cases $k = r$ of Theorem 2 (i) and (iii).

To prove Theorem 8 we observe that, if $\sum a_n$ is summable $\{\mathbf{C}\}_k$, then it is summable $\{C\}_k$ [Theorem 6]. If also it is bounded $\{C, \alpha+1\}_k$, then it is summable $\{C, \beta\}_k$ for every $\beta > \alpha+1$ [Theorem 5], and Theorem 8 follows from this and Theorem 7.

To prove Theorems 10 and 11, it is enough, by Theorems 7 and 8, to show that the hypotheses imply summability (C). If $\sum a_n$ is summable $\{C, \alpha+1\}_k$, it is summable $\{A\}_k$. If also it is summable $\{\mathbf{A}\}_k$, it is summable (A), by Theorem 9 (i). Since summability $\{C, \alpha+1\}_k$ also implies that $\tau_n^\beta = o(1)$ for some β, and since this together with summability (A) implies summability (C),† Theorem 10 now follows.

In the proof of Theorem 11 it is enough (by the analogue of Theorem 6 for 'boundedness') to deal with the case in which $\sum a_n$ is bounded $\{C, \alpha+1\}_k$. Then it is bounded $\{A\}_k$. If also it is summable $\{\mathbf{A}\}_k$ and $k > 1$, it is summable (A), by Theorem 9 (ii). Since boundedness $\{C, \alpha+1\}_k$ also implies that $\tau_n^\beta = O(1)$ for some β, and since this together with summability (A) implies summability (C),‡ this proves the case $k > 1$ of the theorem. If $k = 1$, then we have summability $\{\mathbf{A}\}_r$ for every finite r. Since boundedness $\{C, \alpha+1\}_1$ implies boundedness $\{C\}_r$ for every finite r [Theorem 2 (iii)], and so also boundedness $\{A\}_r$, the case $k = 1$ follows from the case $k > 1$. This completes the proof.

12. We prove next the negative results mentioned in § 5. It is evident that the results for summabilities $\{\mathbf{C}\}_k$ and $\{\mathbf{A}\}_k$ imply those for summabilities $\{C\}_k$ and $\{A\}_k$, and we may therefore restrict ourselves to the former. We have then to prove seven results and we take these in turn.

(*a*) *Summability* $\{\mathbf{C}, \alpha\}_k$ *does not imply summability* $\{\mathbf{C}, \beta\}_r$ *for* $1 \leqslant r \leqslant k$ *and* $-1 < \beta < \alpha$. Let $\beta < \lambda < \alpha$, and take $a_n = (-1)^n E_n^\lambda$.§ Then

† This is, of course, the case $k = \infty$ of Theorem 10, which we take as known.

‡ This is the case $k = \infty$ of Theorem 11.

§ This example is due to Glatfeld (9).

[cf. (**10**) 138] $\sigma_n^\alpha \to 2^{-\lambda-1}$ as $n \to \infty$, while

$$(-1)^n \sigma_n^\beta \sim 2^{-\beta-1} E_n^\lambda / E_n^\beta \sim A(\beta,\lambda) n^{\lambda-\beta},$$

so that $\sum a_n$ is summable $\{\mathrm{C},\alpha\}_k$ to the sum $2^{-\lambda-1}$, but is not summable $\{\mathrm{C},\beta\}_r$.

(*b*) *Summability* $\{\mathrm{C},\alpha\}_1$ *does not imply summability* $\{\mathrm{C},\beta\}_r$ *for* $1 < r < \infty$ *and* $-1 < \beta \leqslant \alpha+1-1/r$. By Theorems 2, 3, 7, it is enough to prove that summability $\{\mathrm{C},\alpha\}_1$ does not imply summability $\{C,\alpha+2-1/r\}_r$ for $1 < r < \infty$.

Write $l_p = 2^{2^p}$, and take $\tau_n^{\alpha+1} = p^{-2} l_p$ if $n = l_p$ $(p = 1,2,\ldots)$, and $\tau_n^{\alpha+1} = 0$ otherwise. Then

$$\sum_1^\infty n^{-1} |\tau_n^{\alpha+1}| = \sum_1^\infty p^{-2} < +\infty,$$

so that $\sum a_n$ is summable $|C,\alpha+1|$. Hence $\sum a_n$ is both summable $(C,\alpha+1)$ and summable $\{C,\alpha+1\}_1$, so that, by Theorem 7, it is summable $\{\mathrm{C},\alpha\}_1$ [cf. § 14].

On the other hand, if $n = l_p+t$, where $0 \leqslant t \leqslant l_p$, and if $\nu = l_p$, then

$$\frac{E_{n-\nu}^{-1/r} E_\nu^{\alpha+1} \tau_\nu^{\alpha+1}}{E_n^{\alpha+2-1/r}} \geqslant A(r,\alpha)(t+1)^{-1/r} l_p^{1/r} p^{-2},$$

whence, for such n,

$$\tau_n^{\alpha+2-1/r} = \frac{1}{E_n^{\alpha+2-1/r}} \sum_{\nu=1}^n E_{n-\nu}^{-1/r} E_\nu^{\alpha+1} \tau_\nu^{\alpha+1} \geqslant A(r,\alpha)(t+1)^{-1/r} l_p^{1/r} p^{-2}.$$

Hence

$$\sum_{n=l_p}^{2l_p} |\tau_n^{\alpha+2-1/r}|^r \geqslant A(r,\alpha) p^{-2r} l_p \sum_0^{l_p} (t+1)^{-1} \geqslant A(r,\alpha) p^{-2r} l_p \log l_p$$
$$\geqslant A(r,\alpha) p^{-2r} 2^p l_p \geqslant A(r,\alpha) l_p,$$

so that $\sum a_n$ is not summable $\{C,\alpha+2-1/r\}$ for $r > 1$.

(*c*) *Summability* $\{\mathrm{C},\alpha\}_k$ *does not imply summability* $\{\mathrm{C},\beta\}_r$ *for*

$$1 < k < r < \infty \quad \textit{and} \quad -1 < \beta < \alpha+1/k-1/r.$$

This is an immediate consequence of Theorem 3 and (*d*).

(*d*) *Summability* $\{\mathrm{C},\alpha\}_k$ *does not imply summability* (C,β) *for* $1 < k < \infty$ *and* $-1 < \beta \leqslant \alpha+1/k$.† It is evidently enough to show that summability $\{\mathrm{C},\alpha\}_k$ does not imply summability $(C,\alpha+1/k)$ for $k > 1$. Let $1 < \lambda < k$, let

$$\sigma_n^\alpha = \left\{\frac{2^p}{\log\log p\,(2^p-n+2)\log^\lambda(2^p-n+2)}\right\}^{1/k} \quad \text{if} \quad 2^p-p < n \leqslant 2^p,$$

† This is stated without proof by Kuttner (**16**) in the case $\alpha = 0$.

where p runs through all integral values for which $\log\log p \geqslant 1$, and let $\sigma_n^\alpha = 0$ otherwise. It is easy to see that $\sum a_n$ is summable $\{C,\alpha\}_k$. On the other hand, if $m = 2^N$,

$$\sigma_m^{\alpha+1/k} \geqslant \frac{1}{E_m^{\alpha+1/k}} \sum_{\nu=m-N+1}^{m} E_{m-\nu}^{1/k-1} E_\nu^\alpha \sigma_\nu^\alpha \geqslant \frac{A(k)}{(\log\log N)^{1/k}} \sum_{s=2}^{N+1} \frac{1}{s \log^{\lambda/k} s},$$

and this last expression tends to ∞ with N since $\lambda/k < 1$.

(*e*) *Summability* $\{C,\alpha\}_1$ *does not imply summability* (C,β) *for*

$$-1 < \beta < \alpha+1.\dagger$$

Here we have only to take $\sigma_n^\alpha = 2^p/\log p$ if $n = 2^p$ $(p = 2,3,\ldots)$ and 0 otherwise, and argue as in (*d*).

(*f*) *Summability* $\{A\}_k$ *does not imply summability* $\{C,\beta\}_r$ *for any* k, r, *and* β. This is a consequence of Theorem 4 and (*g*).

(*g*) *Summability* $\{A\}_k$ *does not imply summability* $\{A\}_r$ *for*

$$1 \leqslant k < r \leqslant \infty.$$

We prove here a stronger result which we require later, namely that the condition

$$\int_0^1 \frac{|\phi(x)|^k}{1-x}\,dx < \infty$$

does not imply that

$$\int_0^R \frac{|\phi(x)|^r}{(1-x)^2}\,dx = o\left(\frac{1}{1-R}\right)$$

for $1 \leqslant k < r < \infty$. Let λ satisfy the conditions $\lambda > 1$ and $2k/r < \lambda < 2$, and let a_n be the nth Taylor coefficient of the function

$$\phi(x) = (1-x)^{-\lambda/k} \exp\{-(1-x)^{-4} \sin^2(\pi/(1-x))\}.$$

Write also $\quad \psi(t) = \phi(1-1/t) = t^{\lambda/k} \exp(-t^4 \sin^2 \pi t).$

Then we have to prove that

$$\int_1^\infty t^{-1}\{\psi(t)\}^k\,dt < \infty, \tag{12.1}$$

and that

$$\int_1^T \{\psi(t)\}^r\,dt \neq o(T) \tag{12.2}$$

as $T \to +\infty$.

Consider first the proof of (12.1). In $n-\frac{1}{2} \leqslant t \leqslant n+\frac{1}{2}$ we have

$$\sin^2 \pi t = \sin^2 \pi(t-n) \geqslant 4(t-n)^2,$$

† This also is stated by Kuttner (16).

so that

$$t^{-1}\{\psi(t)\}^k = t^{\lambda-1}\exp(-kt^4\sin^2\pi t) \leqslant t^{\lambda-1}\exp(-4kt^4(t-n)^2) = \chi(t), \quad \text{say.}$$

Choose now ϵ so that $0 < \epsilon < 2-\lambda$. Then in

$$(n-\tfrac{1}{2})^{-(2-\epsilon)} \leqslant |t-n| \leqslant \tfrac{1}{2}$$

we have
$$(t-n)^2 \geqslant t^{-4+2\epsilon},$$

so that
$$\chi(t) \leqslant t^{\lambda-1}\exp(-4kt^{2\epsilon}),$$

while in
$$0 \leqslant |t-n| \leqslant (n-\tfrac{1}{2})^{-(2-\epsilon)}$$

we have $\chi(t) \leqslant t^{\lambda-1}$. Hence

$$\int\limits_{n-\frac{1}{2}}^{n+\frac{1}{2}} \chi(t)\,dt \leqslant \frac{2(n+\frac{1}{2})^{\lambda-1}}{(n-\frac{1}{2})^{2-\epsilon}} + \int\limits_{n-\frac{1}{2}}^{n+\frac{1}{2}} t^{\lambda-1}\exp(-4kt^{2\epsilon})\,dt,$$

and (12.1) follows by addition, since $\lambda < 2-\epsilon$.

Consider now (12.2). In

$$0 \leqslant |t-n| \leqslant (n+1)^{-2}$$

we have
$$\sin^2\pi t = \sin^2\pi(t-n) \leqslant \pi^2(t-n)^2 \leqslant \pi^2 t^{-4},$$

so that

$$\{\psi(t)\}^r = t^{r\lambda/k}\exp(-rt^4\sin^2\pi t) \geqslant A(r)t^{r\lambda/k} \geqslant A(r)(n-1)^{r\lambda/k}.$$

Hence, for any integer $N \geqslant 2$,

$$\int\limits_{1}^{N+1} \{\psi(t)\}^r\,dt \geqslant A(r)\sum_{2}^{N} \frac{(n-1)^{r\lambda/k}}{(n+1)^2} \geqslant A(r)N^{r\lambda/k-1} \neq o(N),$$

and this proves (12.2).

13. There remain now the negative results mentioned in § 6, and these are almost trivial. To show that summability $\{\mathbf{A}\}_k$ does not imply summability $\{A\}_r$ for any k and r, it is sufficient to show that summability (A) does not imply summability $\{A\}_1$, and to prove this we have only to take $\phi(x) = (1-x)\sin\{\pi/(1-x)^3\}$.

To prove that summability $\{C,\alpha\}_k$ or $\{A\}_k$ does not imply summability $\{\mathbf{C},\beta\}_r$ or $\{\mathbf{A}\}_r$ for any k, r, α, β, it is enough to prove that the relation $\tau_n^\alpha = o(1)$ does not imply summability $\{\mathbf{A}\}_1$. To prove this take $\tau_n^\alpha = (\log n)^{-1}$ for $n \geqslant 2$ and $\tau_1^\alpha = 0$. Then

$$x\phi'(x) = (1-x)^\alpha \sum_{2}^{\infty} E_n^\alpha(\log n)^{-1}x^n \sim A(\alpha)(1-x)^{-1}\left(\log\frac{1}{1-x}\right)^{-1},$$

so that
$$\phi(x) \sim A(\alpha)\log\log\frac{1}{1-x},$$

and $\sum a_n$ is not summable $\{\mathbf{A}\}_1$.

14. Relations between strong and absolute summabilities. In some recent papers (**4, 7**) the author has investigated absolute summability with index k. This has similar properties in many respects to strong summability with index k,† and it is natural to inquire whether there are any relations connecting one with the other. We recall that a series $\sum a_n$ is absolutely summable (C,α) with index k, or summable $|C,\alpha|_k$, where $k \geqslant 1$ and $\alpha > -1$, if the series

$$\sum n^{k-1}|\sigma_n^\alpha - \sigma_{n-1}^\alpha|^k = \sum n^{-1}|\tau_n^\alpha|^k$$

converges, and that it is summable $|A|_k$ if the integral

$$\int\limits_0^1 (1-x)^{k-1}\,|\phi'(x)|^k\,dx$$

is finite. The relations

$$\tau_n^\alpha = o(1) \quad \text{and} \quad (1-x)\phi'(x) = o(1)$$

may be regarded as the cases $k = \infty$ of these properties, so that summability $|C,\alpha|_\infty$ is identical with summability $\{C,\alpha\}_\infty$ and summability $|A|_\infty$ is identical with summability $\{A\}_\infty$. Summability $|C,\alpha|_1$ and summability $|A|_1$ are, of course, summability $|C,\alpha|$ and summability $|A|$, respectively.

It is evident that summability $|C,\alpha|_k$ implies summability $\{C,\alpha\}_k$ and so also summability $\{C,\beta\}_r$ for all r and appropriate β. Further, summability $|A|_k$ implies summability $\{A\}_k$ and so also summability $\{A\}_r$ for $r \leqslant k$. On the other hand, we have:

(i) *Summability $|A|_k$ does not imply summability $\{A\}_r$ for $r > k \geqslant 1$.*‡
In the opposite direction we have also

(ii) *Summability $\{C,\alpha\}_k$ or $\{A\}_k$ does not imply summability $|C,\beta|_r$ or $|A|_r$ for any finite k, r and any α, β.*

Again, it is evident that summability $|C,\alpha|_1$ implies summability (C,α). Since it also implies summability $\{C,\alpha\}_1$, it follows from Theorem 7 that it implies summability $\{C,\alpha-1\}$ for $\alpha > 0$. Further, by Theorem 2, summability $\{C,\alpha\}_1$ implies summability $\{C,\beta\}_r$ for $1 < r < \infty$, $\alpha > -1$, and $\beta > \alpha+1-1/r$. Hence, again by Theorem 7, summability $|C,\alpha|_1$ implies summability $\{C,\beta\}_r$ for $r > 1$ and $\beta > \alpha-1/r > -1$. It is also evident that summability $|A|_1$ implies summability (A) and so also summability $\{A\}_r$ for every r. And at the other extreme, summability $\{C,\alpha\}_\infty$ implies summability $|C,\alpha+1|_\infty$.

† Compare (**8**), equations (5.1) and (5.3), (5.4) and (5.8), (5.11) and (5.13).

‡ This disposes of a question left open in (**4**), namely whether summability $|A|_k$ implies summability $|A|_r$ for $1 < k < r$.

These, however, are the only implications between these forms of summability. More precisely, we have the following results.

(iii) *Summability* $|C,\alpha|_k$ *or* $|A|_k$ *does not imply summability* $\{\mathbf{C},\beta\}_r$ *or* $\{\mathbf{A}\}_r$ *for any* $k > 1$ *and any* r, α, β.

(iv) *Summability* $|C,\alpha|_1$ *does not imply summability* $\{\mathbf{C},\beta\}_r$ *for* $1 < r < \infty$ *and* $-1 < \beta \leqslant \alpha - 1/r$.

(v) *Summability* $\{\mathbf{C},\alpha\}_k$ *does not imply summability* $|C,\beta|_r$ *or* $|A|_r$ *for any* k, α, β *and any finite* r.

(vi) *Summability* $\{\mathbf{A}\}_k$ *does not imply summability* $|C,\beta|_r$ *or* $|A|_r$ *for any* k, r, β.

Of the results (i)–(vi), the last follows from a result proved in § 13,† (iv) is proved by the example of § 12 (b) with $\alpha+1$ replaced by α,‡ and (i) is a consequence of the result proved in § 12 (g) with $(1-x)\phi'(x)$ in place of $\phi(x)$. To prove (ii), (iii), and (v) it is evidently enough to prove the following results.

(h) *Summability* $\{C,\alpha\}_\infty$ *does not imply summability* $|A|_r$ *for any finite* r *and any* $\alpha > -1$. Take $\tau_n^\alpha = (\log n)^{-\lambda}$ for $n \geqslant 2$, and $\tau_1^\alpha = 0$, where $0 < \lambda < 1/r$. Then $\tau_n^\alpha = o(1)$, so that $\sum a_n$ is summable $\{C,\alpha\}_\infty$. Also, as in § 13,

$$\phi'(x) \sim A(\alpha,\lambda)(1-x)^{-1}\left(\log\frac{1}{1-x}\right)^{-\lambda}$$

as $x \to 1-$, so that
$$\int_0^1 (1-x)^{r-1}|\phi'(x)|^r\,dx$$
is divergent, i.e. $\sum a_n$ is not summable $|A|_r$.

(i) *Summability* $|C,\alpha|_k$ *does not imply summability* $\{\mathbf{A}\}_1$ *for any* $k > 1$ *and any* α. Take $\tau_n = (\log n)^{-\lambda}$ $(n \geqslant 2)$, $\tau_1^\alpha = 0$ $(1/k < \lambda < 1)$. Then $\sum a_n$ is summable $|C,\alpha|_k$ and, as in § 13,

$$\phi(x) \sim A(\alpha,\lambda)\left(\log\frac{1}{1-x}\right)^{1-\lambda},$$

so that $\sum a_n$ is not summable $\{\mathbf{A}\}_1$.

(j) *Summability* (C,α) *does not imply summability* $|A|_r$ *for any* α *and any finite* r. Here we require the following lemma:

Lemma 4. *Let* a *be a positive integer such that* $a \geqslant 2$ *and let* $c > 0$. *Let also*

$$u_n = (-1)^n(\log n)^{-1}a^{cn}z^{a^n}, \qquad \psi(z) = \sum_2^\infty u_n,$$

† Since summability $|C,\beta|_r$ implies summability $|A|_r$.

‡ This result disposes of a further question left open in (4), namely whether summability $|C,\alpha|_1$ implies summability $|C,\alpha+1-1/r|_r$ for $r > 1$.

so that ψ is regular in $|z| < 1$. Then there exist constants K and L, depending only on c, such that, if

$$a \geqslant K, \quad n \geqslant L, \quad \exp(-ca^{-n}) \leqslant |z| \leqslant \exp(-\tfrac{1}{2}ca^{-n}),$$

then
$$|\psi(z)| \geqslant \tfrac{1}{2}|u_n| \geqslant \tfrac{1}{2}e^{-c}(\log n)^{-1}a^{cn}.$$

This is proved by an argument identical with one used by Littlewood [(**17**) § 8.5] in similar circumstances, and I therefore omit the proof.

Take now $a_0 = 0$ and

$$\tau_n^\alpha = \frac{(-1)^p a^{p(\alpha+1)}}{E_{a^p}^\alpha \log p} \quad \text{if } n = a^p \ (p = 2, 3, \ldots) \text{ and } \tau_n^\alpha = 0 \text{ otherwise},$$

where a is a positive integer which satisfies the conditions of the lemma with $c = \alpha+1$. By (2.1),

$$\sigma_n^\alpha = \sum_{\nu=1}^{n} \frac{\tau_\nu^\alpha}{\nu} = \sum_{p=2}^{[\log n/\log a]} \frac{(-1)^p a^{p\alpha}}{E_{a^p}^\alpha \log p},$$

so that σ_n^α tends to a limit since $a^{p\alpha}/E_{a^p}^\alpha \log p$ is ultimately decreasing.

On the other hand, by (2.4),

$$x\phi'(x) = (1-x)^\alpha \sum_{p=2}^{\infty} (-1)^p (\log p)^{-1} a^{p(\alpha+1)} x^{a^p}.$$

Writing $x = e^{-t}$, we have now

$$\int_{e^{-1}}^{1} (1-x)^{r-1} |\phi'(x)|^r \, dx \geqslant A(r) \int_{0}^{1} t^{r\alpha+r-1} |\chi(t)|^r \, dt = J, \quad \text{say},$$

where
$$\chi(t) = \sum_{p=2}^{\infty} (-1)^p (\log p)^{-1} a^{p(\alpha+1)} \exp(-ta^p).$$

It follows now from Lemma 4 that, for some integer $q = q(\alpha)$,

$$J \geqslant A(\alpha) \sum_{q}^{\infty} (a^{-n})^{r\alpha+r-1} (\log n)^{-r} a^{rn(\alpha+1)} \tfrac{1}{2}(\alpha+1) a^{-n} = +\infty,$$

as required.

15. I mention in conclusion a further property of the series $\sum a_n$, or of $\phi(x)$, which has connexions with absolute summability.

Let $\phi(z) = \sum a_n z^n$, where z is the complex variable and the series is supposed convergent for $|z| < 1$. Then ϕ is regular in $|z| < 1$, and the mapping $w = \phi(z)$ maps this circle on to a region of the w-plane. The integral which enters into the definition of summability $|A|_1$, namely

$$\int_0^1 |\phi'(x)| \, dx,$$

is then simply the length of the image of the radius from $z = 0$ to $z = 1$

under this mapping, and the summability $|A|_1$ of the series $\sum a_n$ is equivalent to the finiteness of this length.

If we attempt to replace this length by an area, it is natural in this context to consider the area of the image, under the mapping $w = \phi$, of a kite-shaped region with its vertex at $z = 1$, e.g. the region $\Omega(\eta)$ in $|z| < 1$ bounded by the more distant arc of the circle $|z| = \sin\eta$ and the tangents drawn to this circle from the point $z = 1$, where η is a constant such that $0 < \eta < \frac{1}{2}\pi$.† The area of the image of Ω is

$$\iint_{\Omega} |\phi'(z)|^2 \, d\omega, \tag{15.1}$$

where $d\omega$ is the element of area, and we may therefore consider the finiteness of this integral as an analogue of summability $|A|_1$.‡ The presence of the index 2 in this integral suggests that we should associate the integral (15.1) with summability $|A|_2$ rather than with summability $|A|_1$, and this is in fact the case. More generally, we introduce an index k: that is to say, we consider the finiteness of the expression

$$s_k = s_k(\eta) = \left\{ \iint_{\Omega(\eta)} |1-z|^{k-2} |\phi'(z)|^k \, d\omega \right\}^{1/k}.$$

(Since $1 \leqslant |1-z|/(1-|z|) \leqslant A(\eta)$ in Ω, we may replace the factor $|1-z|^{k-2}$ by $(1-|z|)^{k-2}$.)

If this integral is finite, we shall say that $\sum a_n$ possesses the 'property $M(k,\eta)$'. We may also take the relation

$$\lim_{z \to 1 \text{ in } \Omega} (1-z)\phi'(z) = 0$$

to be the case $k = \infty$ of this property.

The following theorem shows that the property $M(k,\eta)$ is intermediate between summability $|C|_k$ and summability $|A|_k$.

THEOREM 13. *Let $1 \leqslant k \leqslant \infty$. Then*

(i) *if $\sum a_n$ possesses the property $M(k,\eta)$ for any $\eta > 0$, it is summable $|A|_k$,*

(ii) *if $\sum a_n$ is summable $|C|_k$, it possesses the property $M(k,\eta)$ for every $\eta < \frac{1}{2}\pi$,*

(iii) *if $1 \leqslant k < \infty$ and $\alpha > -1$, then for any η such that $0 < \eta < \frac{1}{2}\pi$*

$$A(k,\eta) \int_0^1 (1-x)^{k-1} |\phi'(x)|^k \, dx \leqslant s_k^k(\eta) \leqslant A(k,\alpha,\eta) \sum_1^{\infty} n^{-1} |\tau_n^{\alpha}|^k.$$

† In dealing with the behaviour of $\phi(z)$ as $z \to 1$ we are normally concerned with a path of approach to $z = 1$ which is non-tangential to $|z| = 1$. Any given path of this type lies in Ω provided that η is large enough.

‡ The integral (15.1) has been considered by Marcinkiewicz and Zygmund (**19**)

The cases $k = \infty$ of (i) and (ii) are trivial. The cases $k = 2$ of (i) and the left-hand half of (iii) are due to Marcinkiewicz and Zygmund (**19**), while the general cases of these have been given by the author (**6**).

To prove (ii) (for $k < \infty$) and the right-hand half of (iii),† it is evidently enough to prove the latter. We observe that, for some constant μ depending on η,

$$s_k^k \leqslant \int\limits_0^1 (1-\rho)^{k-2}\rho\, d\rho \int\limits_{-\mu(1-\rho)}^{\mu(1-\rho)} |\phi'(\rho e^{it})|^k\, dt.$$

Now $$\phi'(\rho e^{it}) = (1-\rho e^{it})^\alpha \sum_1^\infty E_n^\alpha \tau_n^\alpha \rho^{n-1} e^{(n-1)it},$$

so that, in $\Omega(\eta)$,

$$|\phi'(\rho e^{it})|^k \leqslant |1-\rho e^{it}|^{k\alpha}\Big(\sum_1^\infty E_n^\alpha |\tau_n^\alpha|^k \rho^{n-1}\Big)\Big(\sum_1^\infty E_n^\alpha \rho^{n-1}\Big)^{k-1}$$
$$\leqslant A(k,\alpha,\eta)(1-\rho)^{\alpha-k+1} \sum_1^\infty E_n^\alpha |\tau_n^\alpha|^k \rho^{n-1}.$$

Since also

$$\int\limits_0^1 (1-\rho)^{\alpha-1}\rho^n\, d\rho \int\limits_{-\mu(1-\rho)}^{\mu(1-\rho)} dt = 2\mu \int\limits_0^1 (1-\rho)^\alpha \rho^n\, d\rho \leqslant A(\alpha,\mu) n^{-\alpha-1},$$

the result now follows.

We have also the following theorem:

THEOREM 14. *Let $1 \leqslant k \leqslant r \leqslant \infty$ and $0 < \mu < \eta < \frac{1}{2}\pi$. Then, if $\sum a_n$ possesses the property $M(k,\eta)$, it possesses the property $M(r,\mu)$. Further,*
$$s_r(\mu) \leqslant A(k,r,\mu,\eta) s_k(\eta) \quad (1 \leqslant k \leqslant r < \infty) \tag{15.2}$$
and
$$\sup_{z \in \Omega(\mu)} \{|1-z||\phi'(z)|\} \leqslant A(k,\mu,\eta) s_k(\eta) \quad (1 \leqslant k < \infty). \tag{15.3}$$

The case $k = r$ of (15.2) is evident, while the case $k < r$ follows immediately from the case $k = r$ and (15.3). It is therefore enough to prove that, if $s_k(\eta)$ is finite, where $1 \leqslant k < \infty$, then (15.3) holds and $(1-z)\phi'(z) \to 0$ as $z \to 1$ in $\Omega(\mu)$.

Let ζ be a point of $\Omega(\mu)$ and let $\rho = K|1-\zeta|$, where $K = K(\mu,\eta)$ is chosen so small that the circle $|z-\zeta| \leqslant \rho$ lies in $\Omega(\eta)$ for every ζ of $\Omega(\mu)$. Since $|\phi'|^k$ is subharmonic in $|z| < 1$ [cf. (**20**) § 1.7],

$$|\phi'(\zeta)|^k \leqslant \frac{1}{\pi\rho^2} \int\limits_0^\rho \int\limits_{-\pi}^{\pi} |\phi'(\zeta + Re^{it})|^k R\, dt dR.$$

† These results may be deduced from various integral representations of the series $\sum n^{-1}|\tau_n^\alpha|^k$ given by the author and others. The proof given here, however, is much more elementary.

Since also the circle $|z-\zeta| \leqslant \rho$ is contained in the region $S(\zeta)$ defined by

$$(1-K)|1-\zeta| \leqslant |1-z| \leqslant (1+K)|1-\zeta|, \qquad z \in \Omega(\eta),$$

it follows that

$$|\phi'(\zeta)|^k \leqslant \frac{1}{\pi\rho^2} \iint\limits_{S(\zeta)} |\phi'(1-re^{i\theta})|^k r\, dr d\theta$$

$$\leqslant A(k,\mu,\eta)\rho^{-k} \iint\limits_{S(\zeta)} r^{k-2}|\phi'(1-re^{i\theta})|^k r\, dr d\theta. \tag{15.4}$$

But the integral on the right of (15.4) does not exceed $s_k^k(\eta)$ and tends to 0 as $\zeta \to 1$, and this proves the theorem.

We note in particular the following corollary which follows at once from the main theorem and the left-hand half of Theorem 13 (iii).

COROLLARY. *If* $1 \leqslant k \leqslant r < \infty$, *then, for any* $\eta > 0$,

$$\left\{\int\limits_0^1 (1-x)^{r-1}|\phi'(x)|^r\, dx\right\}^{1/r} \leqslant A(k,r,\eta)s_k(\eta).$$

We observe also that Theorem 14 and § 14 (v) together show that summability $|A|_k$ does not imply the property $M(r,\eta)$ for any r and η.

In conclusion we note that there is a similar analogue of summability (A) involving the region Ω, in which we replace the property

$$\lim_{x \to 1-} \phi(x) = s$$

by the property

$$\lim_{z \to 1 \text{ in } \Omega} \phi(z) = s. \tag{15.5}$$

It is easy to see from (2.3) that, if $\sum a_n$ is summable (C), then (15.5) holds, so that the property (15.5) is intermediate between summability (C) and summability (A). Moreover, for $\alpha > -1$,

$$\sup_{z \in \Omega} |\phi(z)| \leqslant A(\eta,\alpha) \sup_n |\sigma_n^\alpha|.$$

We note also that, if (15.5) holds, then $\sum a_n$ is summable $\{A\}_\infty$, i.e.

$$(1-x)\phi'(x) \to 0$$

as $x \to 1-$.† For $2\pi i \phi'(x)$ is the integral of

$$\{\phi(z)-s\}/(z-x)^2$$

taken round the circle

$$|z-x| = (1-x)\sin\eta,$$

† This remark is due to Littlewood and Paley [(**18**) 58]. We recall (§ 13) that summability (A) does not imply summability $\{A\}_\infty$.

and, since this circle is contained in $\Omega(\eta)$, the integral is $o(1/(1-x))$ as $x \to 1-$. Evidently also

$$\sup_x\{(1-x)|\phi'(x)|\} \leqslant A(\eta)\sup_{z\in\Omega}|\phi(z)|.$$

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ON o-TAUBERIAN THEOREMS

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WIENER remarks in (**3**) that the essential part of the proof of an O-Tauberian theorem is to convert a result about boundedness into one about convergence. In fact it is at this stage of the proof that his general Tauberian theorem is normally applied. For a particular o-type theorem the convergence result can normally be proved by an argument parallel to that which proves the boundedness result. Nevertheless I think that there is some interest in a theorem asserting that a o-type condition sufficient to ensure boundedness will *always* ensure convergence.

Throughout the paper we shall be concerned with a matrix

$$H = [h_{ij}] \quad (1 \leqslant i, j < \infty).$$

A sequence (A_n) $(n \geqslant 1)$ is *summable* to a number N if, firstly, for each $i \geqslant 1$, the series $\sum_j h_{ij} A_j$ converges, to a number which will be written $(HA)_i$, and secondly the sequence $(HA)_i$ converges to N. H is *regular* if every sequence which converges to a number N is also summable to N. I shall assume that H is regular: this is equivalent [see (**1**) or (**2**)] to assuming that

(i) $\sum_j |h_{ij}| < K$ with K independent of i;

(ii) $h_{ij} \to 0$ as $i \to \infty$, for each j;

(iii) $\sum_j h_{ij} \to 1$ as $i \to \infty$.

We shall also consider a fixed sequence (k_n) $(n \geqslant 1)$ of real numbers k_n such that $k_n > 1$. The Tauberian condition, on a sequence (A_n), will be

$$\text{(i)} \quad a_n = o(k_n^{-1}),$$

where $a_1 = A_1$, $a_n = A_n - A_{n-1}$ $(n > 1)$.

Our final assumption about H and (k_n) is that our sequence-to-sequence transformation can be rewritten as a series-to-sequence transformation. More exactly: let

$$l_{i1} = h_{i1} \quad (i \geqslant 1), \qquad l_{in} = \sum_{j=n}^{\infty} h_{ij} \quad (i \geqslant 1;\, n > 1).$$

We require that, for each i,

$$\sum_j |l_{ij}| < \infty,$$

and that $$(HA)_i \equiv \sum_j l_{ij} a_j.$$
whenever (i) holds and either $(HA)_i$ or $(La)_i$ converges.

The object of the paper is to prove the theorem:

THEOREM. *Let H and (k_n) satisfy the conditions stated above. Suppose that the conditions* (i) *above and*

$$\text{(ii)}\quad (A_n) \textit{ is summable}$$

together imply that (A_n) is bounded. Then (i) *and* (ii) *together imply that (A_n) is convergent.*

The natural tool for the purpose is the following variant of the Banach–Steinhaus Theorem [cf. (1), Chapter V, § 1, Theorems 3, 5]:

LEMMA 1. *Let f_n be a sequence of bounded linear functionals on a Banach space $\mathfrak{B}$. If the sequence $f_n(x)$ is bounded for each x in $\mathfrak{B}$ and convergent for all x in some dense subset $\mathfrak{B}_0$ of $\mathfrak{B}$, then it is convergent for all x in $\mathfrak{B}$.*

To apply Lemma 1 to the present situation three further lemmas are required.

LEMMA 2. *The set $\mathfrak{B}$ of sequences $A = (A_n)$ $(n \geqslant 1)$ satisfying conditions* (i) (*with* 0 *as the limit*) *and* (ii) *is a Banach space in the norm*

$$\text{(iii)}\quad ||A|| = \sup_{1 \leqslant n < \infty} k_n|a_n| + \sup_{1 \leqslant i < \infty} |(HA)_i|.$$

Note. It is only in the proof of this lemma that I use the hypothesis (which may be superfluous) that the transformation can be expressed as a series-to-sequence transformation.

Proof. It is enough to show that $\mathfrak{B}$ is complete. In fact let $(A^{(m)})$ $(m \geqslant 1)$ be a Cauchy sequence; here

$$A^{(m)} = (A_n^{(m)}) \quad (n \geqslant 1).$$

Considering the first part of the norm (iii) we see that there is a sequence $A = (A_n)$ $(n \geqslant 1)$ such that $\lim a_n^{(m)} = a_n$ (uniformly in n). Hence, for each i,
$$(HA^{(m)})_i = \sum_j l_{ij} a_j^{(m)}$$
converges to $$\sum_j l_{ij} a_j = (HA)_i.$$
Considering the second part of the norm (iii) we see that the convergence is uniform in i. Clearly $A = (A_n)$ satisfies conditions (i) and (ii) and is the limit of $A^{(m)}$ in the norm (iii).

LEMMA 3. *Each of the linear functionals $f_n(A) = A_n$ is bounded.*

Proof. Consider the first part of the norm (iii): this gives

$$|f_n(A)| \leqslant ||A|| \sum_1^n k_i^{-1}.$$

Definition. $A = (A_n)$ $(n \geqslant 1)$ will be called a *trivial* sequence if $A_n = 0$ for all n sufficiently large.

LEMMA 4. *Let H be a regular matrix. Let $A = (A_n)$ be bounded, and summable (H) to 0. Let $\sum k_n^{-1}$ be a divergent series of positive numbers.*

Then for each positive ϵ and any integer J there is a trivial sequence A' such that

(*a*) $A'_n = A_n \quad (1 \leqslant n \leqslant J)$;

(*b*) $A'_n = l_n A_n \quad (0 \leqslant l_n \leqslant 1;$ *all* $n)$;

(*c*) $|a'_n| \leqslant |a_n| + \frac{1}{3}\epsilon/k_n \quad$ (*all* n);

(*d*) $\sup_i |(HA)_i - (HA')_i| < \epsilon$.

Proof. For each i there are integers j_1, j_2 (not uniquely determined) such that

$$\sum_{j=1}^{j_1} |h_{ij}| + \sum_{j=j_2}^{\infty} |h_{ij}| < \frac{\epsilon}{6 \sup |A_j|}. \tag{A}$$

We can assume that $j_1, j_2 \to \infty$ as $i \to \infty$. All this follows from the regularity of H. I call h_{ij} *insignificant* if $j \leqslant j_1(i)$ or $j \geqslant j_2(i)$.

Let I_0 be an integer such that

$$|(HA)_i| = \Big| \sum_{j=1}^{\infty} h_{ij} A_j \Big| < \tfrac{1}{6}\epsilon, \tag{B}$$

for $i \geqslant I_0$. Let J_0 be an integer greater than J and also large enough for h_{ij} to be insignificant if $i \leqslant I_0$ and $j \geqslant J_0$. Put

$$l_n = 1 \quad (1 \leqslant n \leqslant J_0). \tag{C}$$

Define
$$\delta = \epsilon \Big/ \Big(3 \sup_i \sum_j |h_{ij}| \sup_j |A_j|\Big).$$

Now consider successive positive integers m, defining integers I_m and J_m and the numbers l_n $(J_{m-1} < n \leqslant J_m)$ as follows.

If $1 - m\delta > 0$ and the integer J_{m-1} is defined, let I_m be large enough for h_{ij} to be insignificant if $j \leqslant J_{m-1}$ and $i \geqslant I_m$. Let J_m be large enough for h_{ij} to be insignificant if $i \leqslant I_m$ and $j \geqslant J_m$ and for

$$\sum_{J_{m-1}+1}^{J_m} \frac{\epsilon}{3k_n \sup |A_p|}$$

to be at least δ. Then l_n can be made to decrease from $1-(m-1)\delta$ at $n = J_{m-1}$ to $1 - m\delta$ at $n = J_m$ by differences of at most $\epsilon/(3k_n \sup |A_p|)$.

For the integer $m = M$ for which $1 - m\delta \leqslant 0 < 1-(m-1)\delta$ the words 'to $1-m\delta$' in the above paragraph should be replaced by the words 'to 0'. Put $l_n = 0$ $(n \geqslant J_M)$. No values of m greater than M need be considered.

Now define $A'_n = l_n A_n$. Clearly A' is a trivial sequence, and (a) and (b) are satisfied. We shall have

$$|a'_n| = |\Delta A'_{n-1}| = |l_n \Delta A_{n-1} + A_{n-1}\Delta l_{n-1}| \leqslant |a_n| + \tfrac{1}{3}\epsilon/k_n,$$

proving (c).

For $1 \leqslant i \leqslant I_0$, the definition of J_0 shows that

$$\Big|\sum_1^\infty h_{ij}(A_j - A'_j)\Big| = \Big|\sum_{J_0+1}^\infty h_{ij}(1-l_j)A_j\Big| \leqslant \sum_{J_0+1}^\infty |h_{ij}|.|A_j| < \tfrac{1}{6}\epsilon < \epsilon.$$

For $I_m \leqslant i \leqslant I_{m+1}$ $(0 \leqslant m \leqslant M)$, where we interpret I_{M+1} and J_{M+1} as ∞, we have

$$\begin{aligned}\Big|\sum_1^\infty h_{ij}(A_j - A'_j)\Big| &= \Big|\sum_1^\infty h_{ij}(1-l_j)A_j\Big| \\ &\leqslant \Big|\sum_1^\infty h_{ij}\, m\delta A_j\Big| + \Big|\Big(\sum_1^{J_{m-1}} + \sum_{J_{m+1}}^\infty\Big) h_{ij}(1-m\delta-l_j)A_j\Big| + \\ &\qquad + \Big|\sum_{J_{m-1}+1}^{J_{m+1}-1} h_{ij}(1-m\delta-l_j)A_j\Big| \\ &\leqslant \Big|\sum_1^\infty h_{ij}A_j\Big| + \Big(\sum_1^{J_{m-1}} + \sum_{J_{m+1}}^\infty\Big)|h_{ij}|.|A_j| + \delta\sum_{J_{m-1}}^{J_{m+1}} |h_{ij}|.|A_j| \\ &\leqslant \tfrac{1}{3}\epsilon + \tfrac{1}{3}\epsilon + \tfrac{1}{3}\epsilon = \epsilon,\end{aligned}$$

by (A) and (B).

This proves (d), completing the proof of Lemma 4.

Proof of the theorem. Consider any sequence A satisfying the conditions (i) and (ii) of the theorem. Since the transformation is regular, it will be sufficient to consider the case where A_n is summable to 0 and to show that A_n is convergent to 0; and, since the theorem is trivial if $\sum k_n^{-1}$ converges, I will assume that it diverges. We apply Lemma 1 with $\mathfrak{B}$ and f_n as in Lemmas 2 and 3; $\mathfrak{B}_0$ will be the set of trivial sequences. Given A, by (ii) we can choose J such that

$$k_n|a_n| < \tfrac{1}{3}\epsilon \quad (n > J).$$

Apply Lemma 4, using this J. For the sequence A' given by the lemma we have†

$$\begin{aligned}\|A - A'\| &= \sup_{1\leqslant n<\infty} k_n|a_n - a'_n| + \sup_i |(HA)_i - (HA')_i| \\ &\leqslant \sup_{n>J}(k_n|a_n| + k_n|a'_n|) + \sup_i \Big|\sum_j h_{ij}(A_j - A'_j)\Big| \\ &\leqslant \epsilon + \epsilon,\end{aligned}$$

so that $\mathfrak{B}_0$ is dense. Thus, by Lemma 1, $f_n(A) = A_n$ is convergent.

† It is here alone that the fact that (ii) is a o-condition rather than a O-condition is used.

Second Proof. We can dispense with Lemma 3, and use the 'closed graph' theorem [(**1**), Ch. III § 3 Th. 5] in place of the Banach–Steinhaus theorem if we observe that $\mathfrak{B}$ is also complete in the norm

$$\text{(iv)} \quad \sup k_n|a_n| + \sup|A_n|.$$

This follows from the fact that H is regular.

Since the norm (iii) is less than the norm (iv), the two must be equivalent. Thus (by Lemma 4, applied as in the first proof) any A in $\mathfrak{B}$ can be approximated to—in the norm (iii), hence in the norm (iv), hence uniformly—by elements A' for which A'_n converges. Thus A_n must converge.

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INTEGRABLE-SQUARE SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

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1. THE purpose of this note is to prove a theorem giving the minimal number of integrable-square solutions of ordinary, self-adjoint, linear differential equations with one singular point. Such theorems have so far been given only for real-valued differential operators of even order. The first such theorem, for the second-order case, was given by Hermann Weyl [see (**2**)], a proof of which may also be found in the book by E. C. Titchmarsh on eigenfunction theory [(**1**) Chapter II]. The general even-order case, for real-valued equations, has been discussed by K. Kodaira in (**4**).

The theorem given here applies to all possible forms of self-adjoint differential operators, of odd or even order, real or complex-valued. The result obtained is best-possible. The method of proof uses the limit properties of sequences of Gram matrices, given by the author in (**6**), and does not depend on the use of the 'singular surfaces' as discussed in (**1**), (**2**), and (**4**). However, the resulting theorem is one of existence only and gives no further information about the nature of these integrable solutions. In a later note it is hoped that certain other properties of these solutions will be given.

2. As far as possible the notations used in the book on ordinary differential equations by E. A. Coddington and N. Levinson (**3**) will be adopted here.

The symbol L_n will be used to denote a differential operator of order n $(n \geqslant 1)$. We shall be concerned with operators defined over the half-line $0 \leqslant x < \infty$† which are regular over every closed interval $[0, X]$ $(X > 0)$, i.e., if $y(x)$ possesses n differential coefficients, then‡

$$L_n y \equiv a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_n(x)y, \tag{2.1}$$

where the $n+1$ functions $a_r(x)$ $(0 \leqslant r \leqslant n)$, complex-valued in general, are continuous in $[0, X]$ and

$$a_0(x) \neq 0 \quad (0 \leqslant x \leqslant X) \tag{2.2}$$

† The origin is chosen as the left-hand end-point merely for convenience.

‡ The symbol $y^{(r)}$ will denote the rth differential coefficient of y with respect to x, including the case $r = 1$.

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for all $X > 0$. Thus over the half-line the operator L_n has one singular point, at infinity.

The results obtained would hold equally well in the case when L_n is defined over the finite interval $[0, a)$ (say) with a singularity at $x = a$.

The differential operator adjoint to L_n above is denoted by L_n^+ and [see (**3**) 84] defined as†

$$L_n^+ y \equiv (-1)^n\{\bar{a}_0(x)y\}^{(n)}+(-1)^{n-1}\{\bar{a}_1(x)y\}^{(n-1)}+...+\bar{a}_n(x)y; \quad (2.3)$$

this definition clearly demands further differential properties of the functions $a_r(x)$ which are assumed to hold.

All the operators in this note are taken to be self-adjoint, i.e.

$$L_n = L_n^+ \quad (0 \leqslant x < \infty). \quad (2.4)$$

In this case L_n takes the canonical form [see (**3**) 204 Exercise 14]

$$L_n y \equiv i^n q_0[...\{q_0(q_0 y)^{(1)}\}^{(1)}...]^{(1)}+i^{n-1}q_1[...\{q_1(q_1 y)^{(1)}\}^{(1)}...]^{(1)}+...+$$
$$+i^2 q_{n-2}\{q_{n-2}(q_{n-2} y)^{(1)}\}^{(1)}+iq_{n-1}(q_{n-1} y)^{(1)}+q_n y, \quad (2.5)$$

where the functions $q_r(x)$ satisfy the following conditions, for $0 \leqslant r \leqslant n$ and $0 \leqslant x < \infty$:

(i) q_r is differentiable at least r times,

(ii) $(q_r)^{n+1-r}$ is real-valued,

(iii) $a_0(x) = i^n\{q_0(x)\}^{n+1}$. (2.6)

From the hypothesis (2.2) and the condition (2.6) it follows that $q_0(x)$ is non-vanishing, and, since, from (ii) above, q_0^{n+1} is real-valued, it may be supposed, without loss of generality, that

$$\{q_0(x)\}^{n+1} > 0 \quad (0 \leqslant x < \infty). \quad (2.7)$$

This note is concerned with the differential equation

$$L_n y = \lambda y \quad (0 \leqslant x < \infty), \quad (2.8)$$

where λ is a complex-valued parameter ($\lambda = u+iv$).

A function $f(x)$ is said to be of 'integrable-square over $[0, \infty)$' or 'to belong to $L^2[0, \infty)$' if

$$\int_0^\infty |f|^2\, dx < \infty.$$

3. The following theorem can now be stated:

THEOREM. *Let L_n be a linear, self-adjoint differential operator of order n, defined over the half-line $0 \leqslant x < \infty$ and regular over $[0, X]$ for all $X > 0$. For any value of the complex parameter λ let $S(\lambda)$ denote the*

† A bar denotes the complex conjugate.

maximum number of linearly independent solutions of the differential equation

$$L_n y = \lambda y \quad (0 \leqslant x < \infty,\ \lambda = u+iv) \tag{2.8}$$

which are of integrable-square over $[0,\infty)$.

If n is even, say $n = 2\nu$ ($\nu \geqslant 1$), then, for all λ such that $v = \operatorname{im}\lambda \neq 0$,

$$S(\lambda) \geqslant \nu. \tag{3.1}$$

If n is odd, say $n = 2\nu-1$ ($\nu \geqslant 1$), then

either (i) *for all strictly complex values of λ*

$$\begin{aligned} S(\lambda) &\geqslant \nu-1 \quad \text{if} \quad v > 0,\\ S(\lambda) &\geqslant \nu \qquad\ \ \text{if} \quad v < 0; \end{aligned} \tag{3.2}$$

or (ii) *for all strictly complex values of λ*

$$\begin{aligned} S(\lambda) &\geqslant \nu \qquad\ \ \text{if} \quad v > 0,\\ S(\lambda) &\geqslant \nu-1 \quad \text{if} \quad v < 0. \end{aligned} \tag{3.3}$$

The above inequalities for $S(\lambda)$ are best-possible.

Notes on the above theorem. It will be shown later that, in the case when n is odd, the inequalities (3.2) hold if the condition (2.7) is satisfied. If the sign convention of (2.7) is reversed, (3.3) holds instead of (3.2). Examples serve to show that the above results do not hold if λ is real; in fact, when this is the case, $S(\lambda)$ can be zero (see § 14).

The following sections contain the proof of this theorem. The main idea of the proof is to obtain information about the characteristic roots of the Gram matrix of a fundamental system of solutions of the differential equation. For this the following lemmas are required.

4. Let α_r ($1 \leqslant r \leqslant n$) be any set of constant complex numbers, not all zero. Let $\phi(x,\lambda)$ be that solution of the differential equation (2.8) which satisfies the following initial conditions at the origin,

$$[\phi^{(r-1)}(x,\lambda)]_{x=0} = \alpha_r \quad (1 \leqslant r \leqslant n).$$

The well-known existence theorems tell us that $\phi(x,\lambda)$ exists uniquely and is not identically zero [see (**3**) Chapter 3]; this result will be used in future without explicit mention.

5. For any square matrix of order n the standard notation will be $[a_{rs}]$ (say), where r indicates rows and s columns. The well-known theorems of matrix theory required in this note will be found in the book by L. Mirsky (**5**).

Let $\phi_s(x)$ ($1 \leqslant s \leqslant n$) be any system of n functions all possessing $n-1$

differential coefficients. The symbol $\Phi(x)$ will denote the matrix $[\phi_s^{(r-1)}(x)]$ $(1 \leqslant r,\, s \leqslant n)$, and the following notation will be used for the Wronskian of the system $\{\phi_s\}$

$$W(\phi_1, \phi_2,..., \phi_n)(x) \equiv \det \Phi(x). \tag{5.1}$$

The well-known properties of W for a system of solutions of (2.8) will be assumed [see (**3**) Chapter 3].

6. The Green's formula for the operator L_n and a suitable pair of functions $u(x)$ and $v(x)$ [see (**3**) 86] is

$$\int\limits_{x_1}^{x_2} \{\bar{v} L_n u - u \overline{L_n v}\}\, dx = [u v](x_2) - [u v](x_1) \quad (0 \leqslant x_1 < x_2 < \infty), \tag{6.1}$$

where $[u v]$ is given explicitly by

$$[u v](x) = \sum_{m=1}^{n} \sum_{\substack{p+q=m-1 \\ p \geqslant 0, q \geqslant 0}} (-1)^p u^{(q)} (a_{n-m} \bar{v})^{(p)} \tag{6.2}$$

and the $a_r(x)$ are given in (2.1).

If the form $[u v]$ is written as

$$[u v](x) = \sum_{r=1}^{n} \sum_{s=1}^{n} B_{rs}(x) u^{(s-1)}(x) \bar{v}^{(r-1)}(x), \tag{6.3}$$

then it can be shown, from (6.2), that, for $x \geqslant 0$,

$$\begin{aligned} B_{rs}(x) &= 0 \quad \text{if} \quad r+s > n+1, \\ B_{rr}(x) &= (-1)^{r-1} a_0(x) \quad \text{if} \quad r+s = n+1. \end{aligned} \tag{6.4}$$

Since L_n is self-adjoint, it is also known [see (**3**) 102 Exercise 25] that the matrix $B(x) \equiv [B_{rs}(x)]$ is skew-hermitian, i.e.†

$$B^* = -B \quad (0 \leqslant x < \infty). \tag{6.5}$$

This implies that

$$[u v](x) = -\overline{[v u]}(x) \quad (0 \leqslant x < \infty), \tag{6.6}$$

so that, in particular, $[u u]$ is always a pure imaginary.

It is clear from (6.4) that

$$\begin{aligned} \det B(x) &= (a_0(x))^n \\ &= \{i^n (q_0(x))^{n+1}\}^n, \quad \text{from (2.6).} \end{aligned} \tag{6.7}$$

Thus from (6.7) and (2.7) it follows that $B(x)$ is non-singular for $0 \leqslant x < \infty$.

7. Using the notation of § 5 we have the lemma:

Lemma 1. *Let $\{\phi_1, \phi_2,..., \phi_n\}$ be any system of n functions all with $n-1$*

† An asterisk denotes the transposed conjugate of a matrix.

differential coefficients. Then the following matrix identity holds†

$$[[\phi_r \phi_s](x)]^T = \Phi(x)^* B(x) \Phi(x) \tag{7.1}$$

and
$$\det_{1 \leqslant r,s \leqslant n} [[\phi_r \phi_s](x)] = \{i^n (q_0(x))^{n+1}\}^n |W(\phi_1, \phi_2, ..., \phi_n)(x)|^2, \tag{7.2}$$

both for $0 \leqslant x < \infty$.

Proof. The first result (7.1) follows from matrix multiplication and the representation (6.3) of $[\phi_r \phi_s](x)$. The identity (7.2) follows on taking determinants of (7.1) and using the result (6.7).

It is of some interest to note that the identity (7.2) can be made to give a proof of the fundamental Kodaira–Weyl identity given in (**4**) 504. It seems to indicate that (7.2) is the required generalization of this identity for the operators of this note.

8. Information about the signature of the matrix $-iB(x)$ is required.‡

LEMMA 2. *Let* $B(x)$ *be the non-singular matrix associated with the operator* L_n *(see § 6). Then the following results hold for all* $x \geqslant 0$.

(i) *if* n *is even,* $n = 2\nu$ $(\nu \geqslant 1)$, *the signature of the hermitian matrix* $-iB(x)$ *is zero*;

(ii) *if* n *is odd,* $n = 2\nu - 1$ $(\nu \geqslant 1)$, *the signature of* $-iB(x)$ *is* $+1$.

Proof. This follows from a result in (**5**) [393 Exercise 33] to the effect that, if the non-singular hermitian form $f(x_1, x_2, ..., x_n)$ of order n vanishes for
$$x_{k+1} = x_{k+2} = ... = x_n = 0,$$
where $2k \leqslant n$, whilst the other variables $x_1, x_2, ..., x_k$ are arbitrary, then the signature t (say) of f satisfies the inequality
$$|t| \leqslant n - 2k. \tag{8.1}$$

If n is even, then the properties (6.4) of the elements of $B(x)$ imply the above condition, with $k = \nu$, of the hermitian form with matrix $-iB(x)$. Thus (i) follows at once from (8.1).

If n is odd, then the properties (6.4) imply that (8.1) holds for the matrix $-iB(x)$ with $k = \nu - 1$: that is, for this matrix, $|t| = 1$. Now from (6.7) it follows, since $n = 2\nu - 1$, that the sign of $\det\{-iB(x)\}$ is $(-1)^{\nu-1}$, and this implies that $t = +1$.

Note. In the case when n is odd, the signature of $-iB(x)$ is dependent on the sign-convention (2.7); if this convention is changed to a negative one, then the signature of $-iB(x)$ becomes -1.

† A superscript T denotes the transpose of a matrix.

‡ It is convenient, for later use, to consider the matrix $-iB$ instead of iB.

LEMMA 3. *Let* $\{\phi_s(x,\lambda)\}$ $(1 \leqslant s \leqslant n)$ *be a system of solutions of the differential equation* $L_n y = \lambda y$ *such that*

$$[W(\phi_1, \phi_2, \ldots, \phi_n)(x,\lambda)]_{x=0} \neq 0; \tag{8.2}$$

then the signature of the matrix $-i[[\phi_r \phi_s](x,\lambda)]$ *is identical with that of the associated matrix* $-iB(x)$, *for all* $x \geqslant 0$.

The hypothesis (8.2) implies that†

$$W(\phi_1, \phi_2, \ldots, \phi_n)(x,\lambda) \neq 0 \quad (0 \leqslant x < \infty),$$

so that from the identity (7.1) and the equivalence theorem for hermitian forms [see (**5**) § 12.6] the required result follows.

9. The following lemma on the representation of skew-hermitian matrices is required:

LEMMA 4. *Let* J *be an arbitrary non-singular skew-hermitian matrix of order* n; *then there exists a fundamental system*† *of solutions, say* $\{\phi_s(x,\lambda)\}$ $(1 \leqslant s \leqslant n)$ *of the differential equation* $L_n y = \lambda y$ *such that the representation*

$$J = [[\phi_r \phi_s](0,\lambda)] \tag{9.1}$$

holds for all λ, *if and only if the hermitian matrices* $-iJ$ *and* $-iB(0)$ *have the same signature.*

Proof. If $-iJ$ and $-iB(0)$ have the same signature, then, since both are non-singular, there is a non-singular matrix $P = [\alpha_{rs}]$ (say) such that [see (**5**) § 12.6]

$$-iJ^T = P^*\{-iB(0)\}P. \tag{9.2}$$

Now define the system $\{\phi_s(x,\lambda)\}$, solutions of $L_n y = \lambda y$, by

$$[\phi_s^{(r-1)}(x,\lambda)]_{x=0} = \alpha_{rs} \quad (1 \leqslant r, s \leqslant n). \tag{9.3}$$

From this, (9.2) and the identity (7.1) for $x = 0$ we obtain

$$-iJ^T = \Phi(0,\lambda)^*\{-iB(0)\}\Phi(0,\lambda) = -i[[\phi_r \phi_s](0,\lambda)]^T,$$

and the required result now follows.

Since P is non-singular, the system $\{\phi_s(x,\lambda)\}$ is fundamental. The initial conditions (9.3) are independent of λ, so that (9.1) holds for all λ.

Finally, if (9.1) holds, then it is clear that $-iJ$ and $-iB(0)$ have the same signature.

10. The following lemma on hermitian matrices is required:

LEMMA 5. *Let* A *and* B *be hermitian matrices of order* n *with*

$$\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n \ddagger \quad \textit{and} \quad \nu_1 \leqslant \nu_2 \leqslant \ldots \leqslant \nu_n$$

† See (**3**) 83.

‡ There should be no confusion between the parameter λ and the characteristic roots λ_r; the latter always occur with a subscript.

as their respective characteristic roots. Let the matrix C be defined as

$$C = A+B \tag{10.1}$$

with characteristic roots $\mu_1 \leqslant \mu_2 \leqslant \ldots \leqslant \mu_n$ (say). Then

$$\lambda_r+\nu_1 \leqslant \mu_r \quad (1 \leqslant r \leqslant n). \tag{10.2}$$

Proof. From the definition (10.1) we have, if I is the unit matrix,

$$C-\nu_1 I = A+B-\nu_1 I,$$

where $B-\nu_1 I$ is clearly positive semi-definite. Thus, from a theorem given by Courant and Hilbert [see (7) 107], the characteristic roots of $C-\nu_1 I$ dominate those of A, i.e.

$$\mu_r-\nu_1 \geqslant \lambda_r \quad (1 \leqslant r \leqslant n).$$

11. In this section, for convenience, results concerned with the theory of Gram matrices are stated [see (**6**)].

Let the system of functions $\{\phi\} \equiv \{\phi_1, \phi_2,..., \phi_n\}$ satisfy the conditions

(i) $\{\phi\}$ is linearly independent over $[0, X]$ for all $X > 0$; (11.1)

(ii) $\phi_r \in L^2[0, X]$ $(1 \leqslant r \leqslant n)$ for all $X > 0$. (11.2)

The Gram matrix of $\{\phi\}$ over $[0, X]$ is well-defined, from (11.2) above, as

$$\Gamma(\phi; X) \equiv \left[\int\limits_0^X \phi_r \overline{\phi_s}\, dx\right] \quad (1 \leqslant r, s \leqslant n). \tag{11.3}$$

It is clear that Γ is hermitian, whilst from (11.1) it follows that Γ is positive definite [see (**6**) § 2].

Let the characteristic roots of $\Gamma(\phi; X)$ be $\lambda_r(X)$ $(1 \leqslant r \leqslant n)$ ordered as

$$0 < \lambda_r(X) \leqslant \lambda_{r+1}(X) \quad (1 \leqslant r \leqslant n-1);$$

then it is known [see (**6**) § 8, (**7**) 107] that, for $X' > X$,

$$0 < \lambda_r(X) \leqslant \lambda_r(X') \quad (1 \leqslant r \leqslant n).$$

This monotonic property of the $\lambda_r(X)$ in X permits the following definition of the integer s [see also (**6**) § 8];

$$\lim_{X\to\infty} \lambda_r(X) \begin{cases} < \infty & (r \leqslant s), \\ = \infty & (r > s). \end{cases} \tag{11.4}$$

Clearly $0 \leqslant s \leqslant n$. This leads to the lemma:

LEMMA 6. *Let the system $\{\phi\}$ satisfy the conditions* (11.1) *and* (11.2) *above. Then the maximum number of linearly independent linear forms, i.e. finite sums*

$$\sum_{r=1}^{n} \alpha_r \phi_r(x) \quad (\alpha_r \textit{ constants, not all zero})$$

of the system $\{\phi\}$ which belong to $L^2[0, \infty)$ is s, defined in (11.4) *above.*

The proof of this result will be found in (**6**) §§ 9–11.

COROLLARY. *If the system $\{\phi\}$ is a fundamental system of solutions of the differential equation $L_n y = \lambda y$, then*

$$S(\lambda) = s. \tag{11.5}$$

It is clear that conditions (11.1) and (11.2) are satisfied for such a system $\{\phi\}$, so that (11.5) follows from the definitions of s and $S(\lambda)$, the latter given in § 3.

12. In this section the proof of the theorem in § 3 will be given for the even-order case, i.e. $n = 2\nu$ ($\nu \geqslant 1$).

Let λ be fixed with
$$v = \operatorname{im} \lambda > 0. \tag{12.1}$$

The symbol I_ν will denote the unit matrix of order ν. Let J denote the skew-hermitian matrix

$$J = i\begin{bmatrix} I_\nu & 0 \\ 0 & -I_\nu \end{bmatrix}; \tag{12.2}$$

it is clear that the signature of $-iJ$ is zero. From Lemma 2 the signature of $-iB(0)$ is also zero, so that by Lemma 4

$$J = [[\phi_r \phi_s](0, \lambda)] \tag{12.3}$$

for a fundamental system of solutions, say $\{\phi_s(x,\lambda)\}$ ($1 \leqslant s \leqslant n$), of the differential equation $L_n y = \lambda y$. This system $\{\phi\}$ is now fixed for the rest of this section.

From Lemma 3 the signature of the non-singular hermitian matrix $-i[[\phi_r \phi_s](X,\lambda)]$ is zero for all $X \geqslant 0$; let the characteristic roots of this matrix be

$$\mu_1(X) \leqslant \mu_2(X) \leqslant \ldots \leqslant \mu_n(X), \tag{12.4}$$

so that, for $X \geqslant 0$,

$$\mu_r(X) < 0 \quad (1 \leqslant r \leqslant \nu), \qquad \mu_r(X) > 0 \quad (\nu+1 \leqslant r \leqslant n). \tag{12.5}$$

From the Green's formula (6.1) we have, for all $X > 0$,

$$[\phi_r \phi_s](X,\lambda) - [\phi_r \phi_s](0,\lambda) = \int\limits_0^X \{\bar{\phi}_s L_n \phi_r - \phi_r \overline{L_n \phi_s}\}\, dx = 2iv \int\limits_0^X \phi_r \bar{\phi}_s\, dx,$$

which gives the matrix identity, in the notation of § 11 and (12.3),

$$\Gamma(\phi;\, X) + (2v)^{-1}\{-iJ\} = (2v)^{-1}\{-i[[\phi_r \phi_s](X,\lambda)]\}.$$

To this we can apply Lemma 5 with $A = \Gamma$ and $B = \frac{1}{2}v^{-1}\{-iJ\}$; it is clear that in the notation of this Lemma and by (12.1) we have $\nu_1 = -\frac{1}{2}v^{-1}$.

In the notations of § 11 and (12.4) above this lemma now gives

$$\lambda_r(X) - \frac{1}{2v} \leqslant \frac{\mu_r(X)}{2v} \quad (1 \leqslant r \leqslant n)$$

and so, from (12.5),

$$\lambda_r(X) \leqslant \tfrac{1}{2}v^{-1} \quad (1 \leqslant r \leqslant \nu;\ X \geqslant 0).$$

Thus
$$\lim_{X \to \infty} \lambda_r(X) < \infty \quad (1 \leqslant r \leqslant \nu)$$

so that, for the definition (11.4),†

$$s \geqslant \nu.$$

Finally, by the corollary to Lemma 6 in § 11,

$$S(\lambda) \geqslant \nu, \tag{3.1}$$

and this proves the theorem in this case.

The proof remains virtually unchanged if (12.1) is replaced by $v = \operatorname{im}\lambda < 0$. The signature of $\tfrac{1}{2}v^{-1}\{-i[[\phi_r \phi_s](X,\lambda)]\}$ is still zero, but ν_1 is now equal to $\tfrac{1}{2}v^{-1}$. Thus in this case

$$\lambda_r(X) \leqslant -\tfrac{1}{2}v^{-1} \quad (1 \leqslant r \leqslant \nu;\ X \geqslant 0),$$

and the required result again follows.

13. In this section the odd-order case of the theorem is discussed, i.e. $n = 2\nu - 1$ $(\nu \geqslant 1)$.

Let λ be fixed with
$$v = \operatorname{im}\lambda > 0. \tag{13.1}$$

Let J denote the non-singular skew-hermitian matrix

$$J = i\begin{bmatrix} I_\nu & 0 \\ 0 & -I_{\nu-1} \end{bmatrix},$$

so that the signature of $-iJ$ is $+1$. Again, from Lemmas 2 and 4,

$$J = [[\phi_r \phi_s](0,\lambda)]$$

for a fundamental system $\{\phi_s\}$ of $L_n y = \lambda y$.

Since, in this case, from Lemma 3, the signature of $-i[[\phi_r \phi_s](X,\lambda)]$ is now $+1$, we have, in place of (12.5),

$$\mu_r(X) < 0 \quad (1 \leqslant r \leqslant \nu - 1), \qquad \mu_r(X) > 0 \quad (\nu \leqslant r \leqslant n).$$

The argument now proceeds, as in the previous section, to give

$$\lambda_r(X) \leqslant \tfrac{1}{2}v^{-1} \quad (1 \leqslant r \leqslant \nu - 1;\ X \geqslant 0), \tag{13.2}$$

which yields the required inequality

$$S(\lambda) \geqslant \nu - 1 \tag{3.2}$$

for the case $v > 0$.

If now $v = \operatorname{im}\lambda < 0$, we use the same matrix J and fundamental system $\{\phi_s\}$ above. However, it is clear that the signature of the matrix

† Some of the remaining $\lambda_r(X)$, i.e. for $r > \nu$, may also be bounded for $X > 0$.

$\frac{1}{2}v^{-1}\{-i[[\phi_r\,\phi_s](X,\lambda)]\}$ is now -1 instead of $+1$, so that, in the application of Lemma 5, (13.2) above is replaced by

$$\lambda_r(X) \leqslant -\frac{1}{2v} \quad (1 \leqslant r \leqslant \nu;\ X \geqslant 0),$$

and this gives the required inequality

$$S(\lambda) \geqslant \nu \tag{3.2}$$

for the case $v < 0$.

If the sign-convention (2.7) is reversed, then, as pointed out in § 8, the signature of $-iB(x)$ is then -1 when n is odd. Such a reversal of sign clearly interchanges the above inequalities for $S(\lambda)$ when $v > 0$ and $v < 0$.

14. To complete the proof of the theorem it remains to show, by means of a suitable example, that the above inequalities for $S(\lambda)$ are best-possible.

The example considered is the generalized Fourier equation

$$L_n y \equiv i^n y^{(n)} = \lambda y \tag{14.1}$$

obtained from the general case (2.5) by putting, for $x \geqslant 0$,

$$q_0(x) = 1, \qquad q_r(x) = 0 \quad (1 \leqslant r \leqslant n).$$

This satisfies the sign-convention (2.7).

Solutions of this equation can be obtained explicitly. Let

$$\lambda = \rho e^{i\phi} \quad (-\pi < \phi \leqslant \pi),$$

then n independent solutions of (14.1) are

$$y_r \equiv \phi_r(x) = \exp\left[-i\rho^{1/n}\exp\left\{i\left(\frac{\phi}{n}+\frac{2r\pi}{n}\right)\right\}x\right] \quad (1 \leqslant r \leqslant n).$$

From this it follows that

$$|\phi_r(x)| = \exp\left[x\rho^{1/n}\sin\left(\frac{\phi}{n}+\frac{2r\pi}{n}\right)\right]. \tag{14.2}$$

Consider the case of even values of n $(= 2\nu)$ and let $v = \operatorname{im}\lambda > 0$, i.e. $\rho > 0$ and $0 < \phi < \pi$. It then follows from the distribution of the values of $(\phi+2r\pi)/n$ that the sine of this angle is strictly positive for ν values of r and strictly negative for the other ν values of r, in the range $1 \leqslant r \leqslant n = 2\nu$. In view of the explicit form (14.2) of $|\phi_r|$ it now follows that $S(\lambda) = \nu$ for (14.1) in this case. If $v < 0$, i.e. $\rho > 0$ and $-\pi < \phi < 0$, the argument can be repeated to show that $S(\lambda)$ again takes the value ν.

When n is odd $(n = 2\nu-1)$, similar reasoning shows that $S(\lambda)$ again

attains the lower bounds. In this case the change in sign of ϕ introduces the different values of $S(\lambda)$ for $v > 0$ and $v < 0$.

Finally, this example (14.1) shows that, in general, no result, similar to that of the Theorem in § 3 can hold when λ takes real values. For $\lambda = 0$ the equation (14.1) has for n independent solutions the system $\{x^r;\ 0 \leqslant r \leqslant n-1\}$, and from this it is clear that $S(0) = 0$.

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GENERATING FUNCTIONS AND ASSOCIATED LEGENDRE POLYNOMIALS

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1. Introduction

LET a sequence of functions $f_n(x)$ satisfy Rodrigues' formula

$$f_n(x) = \frac{1}{n!} D_x^n\{(ax+b)^n F(x)\}, \tag{1}$$

where a and b are constants, not both zero, $F(x)$ is independent of n and differentiable an arbitrary number of times. Then it will be shown that, if a generating function

$$\sum_{n=0}^{\infty} a_n t^n f_n(x) \tag{2}$$

is known for either

$$a_n = {}_{p+1}F_q\left[\begin{matrix} -n, \alpha_1, \alpha_2, \ldots, \alpha_p; \\ \beta_1, \beta_2, \ldots, \beta_q; \end{matrix} v\right] \tag{3}$$

or

$$a_n = \frac{(\alpha_1)_n(\alpha_2)_n \cdots (\alpha_p)_n}{(\beta_1)_n(\beta_2)_n \cdots (\beta_q)_n}, \tag{4}$$

it is automatically known for the other. Also a further result will be shown connecting a generating function of the set f_n with one of the set f_{2n}.

The special Jacobi polynomials $P_n^{(\alpha,-\alpha)}(x)$ are essentially [(1) 81 (8)] associated Legendre polynomials. It will be shown that these satisfy a modified form of equation (1) and that the results of the theorem represented by (2), (3), (4) are applicable.

Use will be made of the result given by Chaundy [(**2**) 62 (i)]

$$(1-t)^{-1}{}_{p+1}F_q\left[\begin{matrix} 1, \alpha_1, \alpha_2, \ldots, \alpha_p; \\ \beta_1, \beta_2, \ldots, \beta_q; \end{matrix} \frac{-xt}{1-t}\right] = \sum_{n=0}^{\infty} t^n {}_{p+1}F_q\left[\begin{matrix} -n, \alpha_1, \alpha_2, \ldots, \alpha_p; \\ \beta_1, \beta_2, \ldots, \beta_q; \end{matrix} x\right] \tag{5}$$

and the parallel result [(**3**) 947 (24)]

$$(1-t)^{-1}{}_{p+2}F_q\left[\begin{matrix} \tfrac{1}{2}, 1, \alpha_1, \alpha_2, \ldots, \alpha_p; \\ \beta_1, \beta_2, \ldots, \beta_q; \end{matrix} \frac{xt^2}{(1-t)^2}\right]$$
$$= \sum_{n=0}^{\infty} t^n {}_{p+2}F_q\left[\begin{matrix} -\tfrac{1}{2}n, \tfrac{1}{2}-\tfrac{1}{2}n, \alpha_1, \alpha_2, \ldots, \alpha_p; \\ \beta_1, \beta_2, \ldots, \beta_q; \end{matrix} x\right]. \tag{6}$$

2. Generating-function equivalence

By (1), the $f_n(x)$ may be represented by

$$f_n(x) = \frac{1}{2\pi i}\int\limits_C \frac{(az+b)^n F(z)\,dz}{(z-x)^{n+1}}, \tag{7}$$

where C is a simple closed contour about $z = x$. In what follows it will be necessary to interchange summation and integration in several places. Where the series involved converge, they converge uniformly, and the interchange is justified. The series need not converge however, and in such cases a divergent generating function is formally obtained.

Then

$$\sum_{n=0}^{\infty} t^n f_n(x)\,{}_{p+1}F_q\left[\begin{matrix}-n, \alpha_1, \alpha_2,\dots, \alpha_p; \\ \beta_1, \beta_2,\dots, \beta_q;\end{matrix}\; v\right]$$

$$= \sum_{n=0}^{\infty} t^n\,{}_{p+1}F_q\left[\begin{matrix}-n, \alpha_1, \alpha_2,\dots, \alpha_p; \\ \beta_1, \beta_2,\dots, \beta_q;\end{matrix}\; v\right]\frac{1}{2\pi i}\int\limits_C \frac{(az+b)^n F(z)\,dz}{(z-x)^{n+1}}$$

$$= \frac{1}{2\pi i}\int\limits_C \frac{F(z)}{z-x}\sum_{n=0}^{\infty}\left[\frac{t(az+b)}{z-x}\right]^n {}_{p+1}F_q\left[\begin{matrix}-n, \alpha_1,\dots, \alpha_p; \\ \beta_1,\dots, \beta_q;\end{matrix}\; v\right]dz$$

$$= \frac{1}{2\pi i}\int\limits_C \frac{F(z)}{z-x}\left\{1-\frac{t(az+b)}{z-x}\right\}^{-1}\times$$

$$\times\,{}_{p+1}F_q\left[\begin{matrix}1, \alpha_1,\dots, \alpha_p; \\ \beta_1,\dots, \beta_q;\end{matrix}\; \frac{-vt(az+b)}{z-x-t(az+b)}\right]dz$$

$$= \sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n(-vt)^n}{(\beta_1)_n\cdots(\beta_q)_n}\,\frac{1}{2\pi i}\int\limits_C \frac{(az+b)^n F(z)\,dz}{\{z-x-t(az+b)\}^{n+1}}$$

$$= (1-at)^{-1}\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\left(\frac{-vt}{1-at}\right)^n\times$$

$$\times\frac{1}{2\pi i}\int\limits_C \frac{(az+b)^n F(z)\,dz}{\{z-(x+tb)/(1-at)\}^{n+1}}$$

$$= (1-at)^{-1}\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\left(\frac{-vt}{1-at}\right)^n f_n\!\left(\frac{x+tb}{1-at}\right). \tag{8}$$

It should be noted in the penultimate line above that the pole at $z = (x+bt)/(1-at)$ can always be placed inside C by taking t small

enough and the result extended by continuation on t. Thus we have obtained the result

$$\sum_{n=0}^{\infty} t^n f_n(x)\,{}_{p+1}F_q\left[\begin{matrix}-n, \alpha_1,\ldots,\alpha_p; \\ \beta_1,\ldots,\beta_q;\end{matrix}\, v\right]$$

$$= (1-at)^{-1}\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\ldots(\alpha_p)_n}{(\beta_1)_n\ldots(\beta_q)_n}\left(\frac{-vt}{1-at}\right)^n f_n\left(\frac{x+tb}{1-at}\right), \quad (9)$$

and, if a generating function is known for either side of (9), it is known for the other.

The parallel result

$$\sum_{n=0}^{\infty} t^n f_n(x)\,{}_{p+2}F_q\left[\begin{matrix}-\tfrac{1}{2}n, \tfrac{1}{2}-\tfrac{1}{2}n, \alpha_1,\ldots,\alpha_p; \\ \beta_1,\ldots,\beta_q;\end{matrix}\, v\right]$$

$$= (1-at)^{-1}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n(\alpha_1)_n\ldots(\alpha_p)_n}{(\beta_1)_n\ldots(\beta_q)_n}\left\{\frac{vt^2}{(1-at)^2}\right\}^n f_{2n}\left(\frac{x+tb}{1-at}\right) \quad (9\,\text{a})$$

can be obtained by following exactly the same procedure as in (8) above but by using (6) instead of (5) in the fourth line of the development.

It should now be noted that

$$f_n(x) = L_n^{(\alpha)}(x)e^{-x}x^{\alpha} = \frac{1}{n!}D_x^n\{x^n(e^{-x}x^{\alpha})\}, \quad (10)$$

$$f_n(x) = \frac{(-1)^n H_n(x)e^{-x^2}}{n!} = \frac{1}{n!}D_x^n e^{-x^2}, \quad (11)$$

$$f_n(x) = \frac{(b)_n(-1)^n(-x)^{-a}}{n!(1-x)^{1-b}}\,{}_2F_1\left[\begin{matrix}-n, a; \\ b;\end{matrix}\, \frac{x-1}{x}\right]$$

$$= \frac{1}{n!}D_x^n\{(1-x)^n(-x)^{-a}(1-x)^{b-1}\}, \quad (12)$$

for the Laguerre, Hermite, and hypergeometric functions. Results of the above nature on (10), (11), (12) have been obtained in (**4**) and (**5**). Equations (9) and (9 a) thus contain these as special cases.

3. A type of Rodrigues' formula for $P_n^{(\alpha,-\alpha)}(y)$

In (**1**) [82 (12)] it is shown that, if

$$\rho = (1-2xt+t^2)^{\frac{1}{2}},$$

then

$$\left\{\frac{1+t+\rho}{1-t+\rho}\right\}^{\alpha}\rho^{-m-1}P_m^{(\alpha,-\alpha)}\{(x-t)/\rho\} = \frac{1}{m!}\sum_{n=0}^{\infty}(-1)^m(-n)_m t^{n-m}P_n^{(\alpha,-\alpha)}(x). \quad (13)$$

The right-hand side can further be written as

$$\frac{1}{m!}D_t^m \sum_{n=0}^{\infty} P_n^{(\alpha,-\alpha)}(x)t^n = \frac{1}{m!}D_t^m\left\{\rho^{-1}\left(\frac{1+t+\rho}{1-t+\rho}\right)^\alpha\right\} \tag{14}$$

by a standard generating function for Jacobi polynomials [(6) 68 (4.45)].

Combine (13) and (14), put $x = 0$, and replace t by y to obtain

$$f_n(y) = G(y)P_n^{(\alpha,-\alpha)}\{-y/\sqrt{(1+y^2)}\} = \frac{1}{n!}D_y^n H(y), \tag{15}$$

where
$$H(y) = \left\{\frac{1+y+\sqrt{(1+y^2)}}{1-y+\sqrt{(1+y^2)}}\right\}^\alpha \Big/ \sqrt{(1+y^2)},$$
$$G(y) = H(y)(1+y^2)^{-\frac{1}{2}n}. \tag{16}$$

This satisfies the requirements of (1) with $a = 0$, $b = 1$, and $F(y) = H(y)$. Thus by (9) and (9a) we have

$$\sum_{n=0}^{\infty} H(y)t^n P_n^{(\alpha,-\alpha)}\{-y/\sqrt{(1+y^2)}\}\,{}_{p+1}F_q\left[\begin{matrix}-n, \alpha_1,\ldots,\alpha_p; \\ \beta_1,\ldots,\beta_q;\end{matrix}\, v\right]$$
$$= \sum_{n=0}^{\infty} H(y+t)\frac{(\alpha_1)_n\ldots(\alpha_p)_n}{(\beta_1)_n\ldots(\beta_q)_n}(-vt)^n P_n^{(\alpha,-\alpha)}[(-y-t)/\sqrt{\{1+(y+t)^2\}}], \tag{17}$$

and

$$\sum_{n=0}^{\infty} H(y)t^n P_n^{(\alpha,-\alpha)}\{-y/\sqrt{(1+y^2)}\}\,{}_{p+2}F_q\left[\begin{matrix}-\frac{1}{2}n, \frac{1}{2}-\frac{1}{2}n, \alpha_1,\ldots,\alpha_p; \\ \beta_1,\ldots,\beta_q;\end{matrix}\, v\right]$$
$$= \sum_{n=0}^{\infty} H(y+t)\frac{(\frac{1}{2})_n(\alpha_1)_n\ldots(\alpha_p)_n(vt^2)^n}{(\beta_1)_n\ldots(\beta_q)_n} P_{2n}^{(\alpha,-\alpha)}[(-y-t)/\sqrt{\{1+(y+t)^2\}}]. \tag{18}$$

4. Some special cases

The right-hand side of (17) is given by known generating functions of the general Jacobi polynomial in several special cases, and thus for these cases the left-hand side is known as well. Write

$$T = \frac{-vt}{\sqrt{\{1+(y+t)^2\}}}, \qquad X = -\frac{y+t}{\sqrt{\{1+(y+t)^2\}}}. \tag{19}$$

Then, by (7),

$$(1-t)^{-1}\,{}_2F_1\left[\begin{matrix}\frac{1}{2}, 1; \\ 1+\alpha;\end{matrix}\, \frac{2t(x-1)}{(1-t)^2}\right] = \sum_{n=0}^{\infty} \frac{n!}{(1+\alpha)_n} P_n^{(\alpha,-\alpha)}(x)t^n \tag{20}$$

yields

$$H(y+t)(1-T)^{-1}\,{}_2F_1\left[\begin{matrix}\frac{1}{2}, 1; \\ 1+\alpha;\end{matrix}\, \frac{2T(X-1)}{(1-T)^2}\right]$$
$$= \sum_{n=0}^{\infty} H(y)t^n P_n^{(\alpha,-\alpha)}\{-y/\sqrt{(1+y^2)}\}\,{}_2F_1\left[\begin{matrix}-n, 1; \\ 1+\alpha;\end{matrix}\, v\right]. \tag{21}$$

The relation (**8**)

$$ {}_0F_1(1+\alpha;\tfrac{1}{2}t(x-1))_0F_1(1-\alpha;\tfrac{1}{2}t(x+1)) = \sum_{n=0}^{\infty} \frac{P_n^{(\alpha,-\alpha)}(x)t^n}{(1+\alpha)_n(1-\alpha)_n} \tag{22} $$

yields

$$ H(y+t)_0F_1(1+\alpha;\ \tfrac{1}{2}T(X-1))_0F_1(1-\alpha;\ \tfrac{1}{2}T(X+1)) $$

$$ = \sum_{n=0}^{\infty} H(y)t^n P_n^{(\alpha,-\alpha)}\{-y/\sqrt{(1+y^2)}\}_1F_2\left[\begin{matrix}-n; \\ 1+\alpha,\ 1-\alpha;\end{matrix}\ v\right]. \tag{23} $$

The result (**3**)

$$ {}_2F_1\left[\begin{matrix}a,\ 1-a; \\ 1+\alpha;\end{matrix}\ \tfrac{1}{2}(1-t-\rho)\right]{}_2F_1\left[\begin{matrix}a,\ 1-a; \\ 1-\alpha;\end{matrix}\ \tfrac{1}{2}(1+t-\rho)\right] $$

$$ = \sum_{n=0}^{\infty} \frac{(a)_n(1-a)_n}{(1+\alpha)_n(1-\alpha)_n} P_n^{(\alpha,-\alpha)}(x)t^n \tag{24} $$

gives nothing new since the corresponding result was given in (**1**).

The use of only the even powers of t on the right-hand sides of (20), (22), (24) yields cases which can be applied to the right-hand side of (18), but none seem to be simple enough to make their inclusion here justified.

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The relation (**8**)

$$ {}_0F_1(1+\alpha; \tfrac{1}{2}t(x-1))\,{}_0F_1(1-\alpha; \tfrac{1}{2}t(x+1)) = \sum_{n=0}^{\infty} \frac{P_n^{(\alpha,-\alpha)}(x)t^n}{(1+\alpha)_n(1-\alpha)_n} \tag{22} $$

yields

$$ H(y+t)\,{}_0F_1(1+\alpha; \tfrac{1}{2}T(X-1))\,{}_0F_1(1-\alpha; \tfrac{1}{2}T(X+1)) $$

$$ = \sum_{n=0}^{\infty} H(y)t^n P_n^{(\alpha,-\alpha)}\{-y/\sqrt{(1+y^2)}\}\,{}_1F_2\left[\begin{matrix}-n; \\ 1+\alpha,\, 1-\alpha;\end{matrix}\; v\right]. \tag{23} $$

The result (**3**)

$$ {}_2F_1\left[\begin{matrix}a,\, 1-a; \\ 1+\alpha;\end{matrix}\; \tfrac{1}{2}(1-t-\rho)\right]{}_2F_1\left[\begin{matrix}a,\, 1-a; \\ 1-\alpha;\end{matrix}\; \tfrac{1}{2}(1+t-\rho)\right] $$

$$ = \sum_{n=0}^{\infty} \frac{(a)_n(1-a)_n}{(1+\alpha)_n(1-\alpha)_n} P_n^{(\alpha,-\alpha)}(x)t^n \tag{24} $$

gives nothing new since the corresponding result was given in (**1**).

The use of only the even powers of t on the right-hand sides of (20), (22), (24) yields cases which can be applied to the right-hand side of (18), but none seem to be simple enough to make their inclusion here justified.

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